14^{th} Feb :

midsem syllabus: 8 problems ~ 15 min per problem
80% Tutorial
20% New

Assignment-2 : Due 19^{th} Feb, submit to TA

$u \in C^2(\mathbb{R}^2)$
↳ twice cont. differentiable on $\mathbb{R}^2$
$u(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Laplacian: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\Delta . u = 0$
$\Rightarrow$ u is harmonic

Existence of harmonic conjugate :

Theorem : If $\Omega \subseteq \mathbb{R}^2$ is a convex open set, and $u \in C^2(\Omega)$, s.t
$u_{xx} + u_{yy} = 0$ on Ω
then $\exists\ v \in C^2(\Omega)$
s.t u, v are harmonic conjugates
($\exists f : \Omega (\subseteq \mathbb{C}) \longrightarrow \mathbb{C}$ is holomorphic with $f = u + iv$)

proof : consider $f(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y)$ for $z = x+iy \in \Omega (\subseteq \mathbb{C})$

(we will find this v by $f = \frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y}$)

f is holomorphic on Ω
as it satisfies the C-R equation
and u_x, u_y, v_x, v_y are continuous

$u_1 = u_x$
$v_1 = -u_y$

$(u_1)_x = u_{xx}$

$(u_1)_y = u_{xy}$

$(v_1)_y = -u_{yy} = u_{xx} = (u_1)_x$

$(v_1)_x = -u_{xy} = -(u_1)_y$

so $u_x = v_y$
$u_y = -v_x$

$\therefore$ f satisfies the C-R equation
and as second partial derivates are cont we have $f \leftarrow$ holomorphic

By the construction of path-integrals

f has a primitive on Ω
so $\exists\ g' = f$
now as Ω is convex, by defining $g(z) = \int_\gamma f(w)dw + u(z_0)$

where γ is the straight line path from $z_0 \in \Omega$ (fixed) to $z \in \Omega$ (variable)

$\uparrow$ just a constant factor

then $g'(z) = f(z)$
now, wlog we may assume $g(z_0) = u(z_0)$

as $\int_{\gamma} f(w)dw + u(z_0) = g(z)$
$\gamma \to$ from z_0 to z

$$\int_{z_0}^{z_0} f(w)dw = 0$$

now, claim : $u = Re(g)$
as $g'(z) = f(z) = \frac{\partial}{\partial x}u - i\frac{\partial}{\partial y}u$
and $g'(z) = \frac{\partial}{\partial x}u - i\frac{\partial}{\partial y}u$

in the x direction :

$$g(z) = u_2 + iv_2$$

$\left(g(z) = u_2 + iv_2 \quad g'(z) = u_x - i\, u_x \quad g'(z) = (v_2)_x + i(v_2)_x \right)$

x-direction $\to g'(z) = (u_2)_x + i(v_2)_x$
y direction $\to g'(z) = \frac{1}{i}\left((u_2)_y + i(v_2)_y\right)$

$$= (v_2)_y - i(u_2)_y$$

and $g'(z) = f(z) = \frac{\partial}{\partial x}u - i\frac{\partial}{\partial y}u$
$= u_x - iu_y$

comparing both :

① — $\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}u_2$

$u_x = (u_2)_x$
$u_y = (u_2)_y$
$\Rightarrow Re(g) = u$

$+\frac{\partial}{\partial y}u = +\frac{\partial}{\partial y}u_2$
② — $\Rightarrow \frac{\partial}{\partial y}u = \frac{\partial}{\partial y}u_2$

and $(u-u_2)_x = (u-u_2)_y = 0$

$\Rightarrow u - u_2 = c$ for some const
at z_0:

$\Rightarrow u(z_0) - u_2(z_0) = c$
$\Rightarrow u(z_0) - g(z_0) = c$
$\Rightarrow c = 0$
$\Rightarrow u = Re(g)$
$= u_2$

imaginary part of $g(z_0) = 0$

(comparing u, v)

let $v = Img(g)$
then this shows existence of $v: \mathbb{R}^2 \to \mathbb{R}$ s.t $u + iv$ is hol
moreover as g is infinitely c-diff we get that so is u: $u: \mathbb{R}^2 \to \mathbb{R}$
(as $\exists g$ s.t $Re(g) = u \Rightarrow Img(g) = v \Rightarrow \Delta \cdot v = 0$ and u, v are harmonic conjugates)

$\Rightarrow$ If $u \in C^2(\Omega)$ for Ω a convex open subset of $\mathbb{R}^2$ and $u_{xx} + u_{yy} = 0$ then $u \in C^\infty(\mathbb{R}^2)$

where $g(z) = \int_{z_0}^{z} f(z)\,dz + u(z_0)$

$$f(z) = \frac{\partial}{\partial x}u - i\frac{\partial}{\partial y}u$$

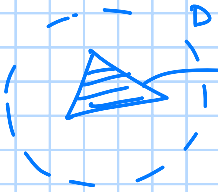
$\downarrow$ holomorphic

Theorm: (Morera's theorm) say that D is a disk, and f is continious on $\operatorname{int}(D)$ s.t every triagle τ lying inside $\operatorname{int}(D)$ we have

$$\int_T f(z)\,dz = 0$$

$\hookrightarrow T = \partial\tau$

then f is holomorphic on $\operatorname{int}(D)$



$\partial\tau = T$

proof:

choose $z_0 \in \operatorname{int}(D)$

define $g(z) = \int_\gamma f(w)\,dw$

$\gamma \leftarrow$ straight line from z_0 to z

now from the proof of Gursat's theorm

$\Rightarrow g(z)$ is holomorphic

$\Rightarrow g'(z) = f(z)$

$\Rightarrow f$ is holomorphic

$\left(\int_z^{z+h} \frac{f}{h} = \int_z^{z+h} \frac{|h|}{h}\gamma \to 0 \text{ as } f \text{ is cont} \right)$

$\Rightarrow g' = f \Rightarrow f$ is

(from similar work in gursat's

$g(z+h) - g(z) = \int_{z_0}^{z+h} f - \int_{z_0}^{z} f = \int_z^{z+h} f \to 0$ as $h \to 0$

$\therefore$ continious)

Theorm: (uniform limit of hol fun is hol) say Ω is open set in $\mathbb{C}$, and $\{f_n\}_{n\geq 1}$ is a sequence of holomorphic fu $f_n : \Omega \to \mathbb{C}$ s.t for any compact set $K \subseteq \Omega$, f_n converges uniformally (to f), then f is holomorphic.

(if $f_n : \mathbb{R} \to \mathbb{R}$

$\{f_n\} \xrightarrow{\text{unif}} f$ then f need not be diff,

$f_n(x) = \sqrt{x^2 + 1/n}$, $n \geq 1$

$f_n \to |x|$ every $K \subseteq \mathbb{R}$

$|x|$ is not diff at $x = 0$)

proof: say $z_0 \in \Omega$, choose small nbd of z_0 say $\{z \mid |z - z_0| < \varepsilon\} \subseteq \Omega$

s.t $D_\varepsilon(z_0) \subseteq \Omega$

since each f_n is holomorphic on the disk

$\int_T f_n(z)\,dz = 0$ for any $T = \partial\tau$ when $\tau \subseteq \operatorname{int}(D_\varepsilon(\varepsilon))$

also $f_n \to f$ uniformly on $D_\varepsilon(z_0)$

$\Rightarrow \lim_{n\to\infty} \int_T f_n(z)\,dz = \int_T \lim_{n\to\infty} f_n(z)\,dz$ (this is by uniform continuity)

$$0 = \int_T f(z)\,dz$$

this is for all triangle in $\operatorname{int}(D_\delta(z_0))$

$\Rightarrow$ By morera's theorem, f is hol on $D_\delta(z_0)$, z_0 is arbitrary

$\Rightarrow f$ is hol on Ω

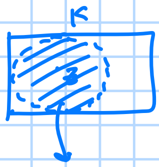
Theorem: Under previous hypothesis, $f_n' \longrightarrow f'$ uniformly on any compact set $K \subseteq \Omega$

proof: Idea is that if $\underset{\text{compact}}{K \subseteq \Omega}$ then

K is closed and bounded

So, $\exists\, \delta > 0$ s.t $\forall\, z \in K$ we have the disk $D_\delta(z) \subseteq \Omega$

this value of δ depends on K, but as K is fixed
$\Rightarrow \delta$ is fixed



$D_\delta(z) \subseteq \Omega$

$\hookrightarrow$ fixed δ as K is closed and bounded

If F is holomorphic on Ω then by cauchys integral formula

$$F'(z) = \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{F(\omega)}{(\omega - z)^2}\,d\omega$$

$C_\delta(z) \longrightarrow$ Boundary of $\partial D_\delta(z)$

$$\Rightarrow |F'(z)| \leq \frac{1}{2\pi} \int_{C_\delta(z)} \frac{|F(\omega)|\,|d\omega|}{|\omega - z|^2}$$

$$\leq \frac{1}{\cancel{2\pi}} \times \frac{\cancel{2\pi}}{\delta} \times \sup_{\omega \in C_\delta(z)} |F(\omega)|$$

$$\leq \frac{1}{\delta} \sup_{\omega \in C_\delta(z)} |F(\omega)|$$

$$\leq \frac{1}{\delta} \sup_{\omega \in K} |F(\omega)|$$

now take $F(z) = f_n(z) - f(z)$

$\Rightarrow F'(z) = f_n'(z) - f'(z)$ (By choosing F so that $f_n - f \longrightarrow 0$)

$$\Rightarrow |f_n'(z) - f'(z)| \leq \sup_{z \in K} \frac{|f_n(z) - f(z)|}{\delta}$$

as $f_n \longrightarrow f$ unif
$\Rightarrow \sup_{\omega \in K} |f_n(\omega) - f(\omega)| \leq B$
can be made as small as we want say $B < \varepsilon\delta$

$\Rightarrow |f_n'(z) - f'(z)| \leq B/\delta \quad \forall\, z \in K \leftarrow$ uniform converge (from def $f_n \to f$)

Note: similarly $f_n^{(k)} \longrightarrow f^{(k)}$ uniformly on $\underset{\text{compact}}{K \subseteq \Omega}$ $\forall\, k \geq 0$

(this is true for all $k \in \mathbb{N}$)

4^{th} March:

Recap: 1) gave $\mathbb{R}^2$ the complex structure
2) we defined notion of C-R equations
3) Power series, exp, log

New focus [
4) holomorphic functions
5) Path integrals

6) Rigidity, goursat theorm, moreras theorm, cauchy integral formula

Isolated singularities:

(fution defined on $\Omega \setminus \{z_0\}$)

say $z_0 \in \Omega \underset{open}{\subset} \mathbb{C}$, and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is continious, and then exist a small nbd U of z_0 s.t $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic

then z_0 is called isolated singularity of f (f is cont, some nhd hol)

Eg: 1) $f(z) = \frac{1}{z}$, $z = 0$ is an isolated singularity

2) $f(z) = \frac{1}{z+1}$ $z = -1$ is an isolated singularity

3) $f(z) = z$ on $\mathbb{C} \setminus \{0\}$ has isolated singulairy at $z = 0$
in this case
$z = 0$ is removable singularity (Removable sing $\Rightarrow$ isolated sing)

Removable singularity:

$z_0 \in \Omega$ is called removable if $\exists g : U \to \mathbb{C}$ s.t $g(z) = f(z)$ for $z \in U \setminus \{z_0\}$
and $g : U \to \mathbb{C}$ is holomorphic

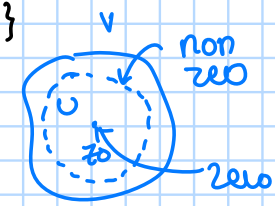
$\hookrightarrow$ this just means we can remove / add z_0

Theorm: say $f : \Omega \to \mathbb{C}$ is hol, Ω is a connected open set, say f is not id 0 ($f \not\equiv 0$) & say $\exists z_0 \in \Omega$ s.t $f(z_0) = 0$

then, $\exists$ ① open nbd U of z_0, $U \subseteq \Omega$
② $g : U \to \mathbb{C}$ is hol s.t $g(z) \neq 0$, $\forall z \in U$

③ $n \in \mathbb{Z}_{>0}$ unique s.t
$f(z) = (z - z_0)^n g(z)$ for $z \in U$

proof: Ω is connected and open & $f \not\equiv 0 \Rightarrow \exists$ nbd V of z_0 s.t $f(z) \neq 0$ for $z \in V \setminus \{z_0\}$



inside V, we can find a small open disk U centerd at z_0 & expand $f(z)$ into power series

$$f(z) = a_0 + a_1(z - z_0) + \cdots$$
$$= \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

as, $z \in U$, $f \not\equiv 0$ on U
we have some $a_k \neq 0$

Say $m \in \mathbb{Z}_{>0}$ is smallest s.t $a_m \neq 0$

then $f(z) = \sum_{k=m} a_k (z - z_0)^k$
$$= (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$$

let $g(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{k-m}$

as power series conv in $|z|<R$, and diff so g is holomorphic

or $g(z)$ is uniform limit of polynomials

as polynomials are hol $\Rightarrow$ uniform limit is hol $\Rightarrow g(z)$ is hol

also $g(z) \neq 0 \ \forall z \in U$ (Reason why g is hol)

for $z = z_0$

$g(z_0) = a_m \neq 0$

for $z \neq z_0$ $f(z) \neq 0$

$\underset{\neq 0}{f(z)} = \underset{\neq 0}{(z-z_0)^m} g(z)$

$\Rightarrow g(z) \neq 0$

for uniqueness, $f(z) = (z-z_0)^m g(z)$
$= (z-z_0)^n h(z)$

wlog $m>n$ then

$\frac{f(z)}{(z-z_0)^n} = (z-z_0)^{m-n} g(z) = h(z)$

for $z = z_0$

$h(z_0) \neq 0$ but $(z_0)^{m-n} g(z_0) = 0$

this is not true

$\Rightarrow$ contradiction

so, $m = n \Rightarrow m$ is unique

Def: This m is called the order of the zero $z = z_0$

if $m=1$ then $z=z_0$ is called simple zero

simple as $f(z) = (z-z_0) g(z)$

Defn: Say $z_0 \in U \underset{\text{open}}{\subseteq} \mathbb{C}$, we call $U \setminus \{z_0\}$ as deleted nbd of z_0

Defn: (Pole of f) f is said to have a pole at a point $z = z_0$ if f is well defined in a deleted nbd $U \setminus \{z_0\}$ of z_0, and ($\exists U \underset{\text{open}}{\subseteq} \mathbb{C}$)

(we don't know f in z_0 only $U \setminus \{z_0\}$)

$$\tilde{f} := \begin{cases} 0 & \text{where } z = z_0 \\ \frac{1}{f(z)} & \text{where } z \neq z_0, z \in U \end{cases}$$

is holomorphic on U

Theorem: If f has a pole at $z_0 \in \Omega \underset{\text{open}}{\subseteq} \mathbb{C}$, then $\exists U \underset{\text{open}}{\subseteq} \Omega$, $z_0 \in U$ a non-vanishing hol function

$h: U \to \mathbb{C}$

and unique $n \in \mathbb{Z}_{>0}$ s.t

$f(z) = (z-z_0)^{-n} h(z)$

$\forall z \in U \setminus \{z_0\}$

proof: use $\tilde{f}$ is hol and theorem (done above)

$\Rightarrow$ open nbd $U \subseteq \Omega$, $z_0 \in U$, $g: U \to \mathbb{C}$

and $\exists m \in \mathbb{Z}_{>0}$ unique (using the theorem) s.t $\tilde{f}(z) = (z-z_0)^m g(z)$ and $g \neq 0 \ \forall z \in U$ (as $\tilde{f}(z)$ is hol and at 0 $\tilde{f}(z_0)=0$)

for $z \neq z_0$ $\frac{1}{f(z)} = (z-z_0)^m g(z)$

$\Rightarrow f(z) = (z-z_0)^{-m} h(z) \ \forall z \in U \setminus \{z_0\}$

$h(z) = \frac{1}{g(z)}$

($h(z)$ is holomorphic as $h(z) = \frac{1}{g(z)}$, $g(z) \neq 0$)

Defn : m above is called order of the pole

so far : 1) f is hol on Ω, $z_0 \in U$ and $f(z_0)=0 \Rightarrow \exists U \subseteq \Omega$
s.t $f(z) = (z-z_0)^m g(z)$
($m>0$, $g \neq 0$, g is hol)

2) If f has pole at $z_0 \in \Omega$

$$\tilde{f} = \begin{cases} 0 & , z = z_0 \\ 1/f & , z \neq z_0 \end{cases}$$

also $f(z) = (z-z_0)^{-n} h(z)$
m = order of 0
n = order of pole

Theorm : say f has a pole of order n at $z=z_0$, then we can write

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \dots \frac{a_{-1}}{(z-z_0)} + G(z)$$

for $z \in U$ some small nbd of z_0, with $a_{-n} \neq 0$
$G(z)$ is holomorphic

proof : prev theorm
$\Rightarrow f(z) = (z-z_0)^{-n} h(z)$
and expand $h(z)$ around $z=z_0$

$$\Rightarrow f(z) = (z-z_0)^{-n} \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

as by definition $h(z)$ is hol

as $h(z_0) \neq 0$
$\Rightarrow a_{-n} \neq 0$

Defn: 1) f is called meromorphic on Ω if $F: \Omega \to \mathbb{C}$ only has isolated singularities which are poles.

(so if z is an isolated singularity then it is a pole)

(Removal or pole of finite order for meromorphic)

2) a_{-1} is called residue of $f(z)$.

Exe : $f(z) = \frac{1}{z}$ at $z=0$

$a_{-1} = 1 \Rightarrow \operatorname{Res}(f,0) = 1$

$g(z) = \frac{1}{z^2}$, $\operatorname{res}(g,0) = 0$

but has pole at 0.

$h(z) = z$, $\operatorname{res}(h,\omega) = 0 \ \forall \omega \in \mathbb{C}$

proposition: If f has pole of order n at $z = z_0$, then

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \left[(z-z_0)^n f(z)\right]$$

proof: $f(z) = (z - z_0)^{-n} h(z)$

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots \frac{a_{-1}}{(z-z_0)} + G(z)$$

$$\Rightarrow (z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0)^1 + \cdots a_{-1}(z-z_0)^{n-1} + (z-z_0)^n G(z)$$

$$\left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z) = (n-1)!\, a_{-1} + \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n G(z)\right]$$

taking $G(z) = \sum_{k=0}^{\infty} b_k (z-z_0)^k$

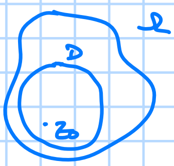
$$\Rightarrow \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n G(z)\right] = 0$$

so $$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} \left[(z-z_0)^n f(z)\right]$$

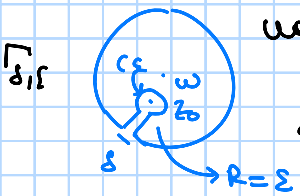
Residue formula:

suppose f has a pole at $z_0 \in \Omega$ $(f: \Omega \to \mathbb{C})$ and say $\exists$ a disk $D \subseteq \Omega$ s.t $z_0 \in \operatorname{int}(D)$ & f is hol on $D \setminus \{z_0\}$ then

$$\frac{1}{2\pi i} \oint_{\partial D} f(z)\, dz = \operatorname{res}(f, z_0)$$



we use the keyhole contour,



we had shown

$$\frac{1}{2\pi i} \int_{\Gamma_{\delta,\varepsilon}} f(w)\, dw = 0$$

and letting $\delta \to 0$

$$\frac{1}{2\pi i} \int_C f(w)\, dw = \frac{1}{2\pi i} \int_{C_\varepsilon} f(w)\, dw$$

C_ε: small circle with centre z_0 Radius ε

use $$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + \underbrace{G(z)}_{\text{holomorphic}}$$

$$\frac{1}{2\pi i} \int_{C_\varepsilon} f(z)\, dz = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{a_{-n}}{(z-z_0)^n} dz + \cdots + \frac{1}{2\pi i} \int \frac{a_{-1}}{(z-z_0)} dz + \frac{1}{2\pi i} \int G(z)\, dz$$

now Cauchy formula tells us that

for $g(z) = a_{-1}$

$$g(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{a_{-1}}{z - z_0} = a_{-1}$$

also $$\frac{1}{2\pi i} \int \frac{a_{-k}}{(z-z_0)^k} = (k-1)!_0 \left. \frac{d^{k-1}}{dz^{k-1}} (a_{-k}) \right|_{z=z_0}$$

$$= 0 \quad \text{for } k \geq 2$$

& $$\frac{1}{2\pi i} \int_{C_\varepsilon} a(z) dz = 0$$

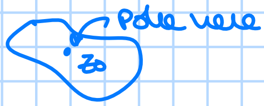
So, $$\frac{1}{2\pi i} \int_C f(z) dz = a_{-1} = \operatorname{Res}(f, z_0)$$

$$\left(\frac{1}{2\pi i} \int_C f(z) dz = \operatorname{Res}(f, z_0) \right)$$

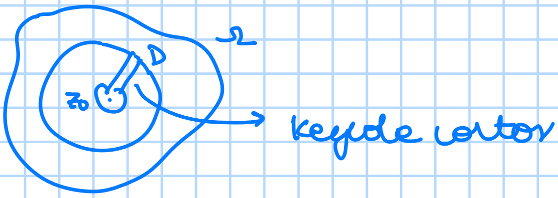
7th Feb :

Recap : say $f : \Omega \to \mathbb{C}$ is hol and has pole at z_0, say $z_0 \in D \subseteq \Omega$, then

$$\frac{1}{2\pi i}\int_{\delta D} f(z)\,dz = \operatorname{Res}(f, z_0) \qquad z_0 \in \operatorname{int}(D)$$

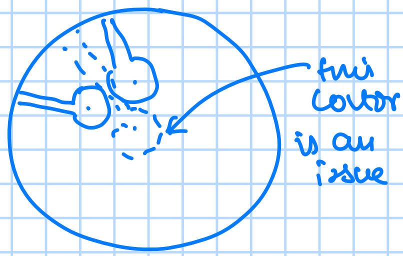


we proved it using contour integration using keyhole contour.



In general if f is holomorphic on Ω, except a "finite" of many points then f may have a pole, say $D \subseteq \Omega$
s.t f is hol on D except for
$z_1, \dots z_n \in \operatorname{int}(D)$ where f has a pole
then

$$\frac{1}{2\pi i}\int_{\delta D} f(z)\,dz = \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$



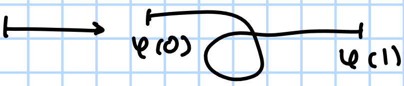
so we can/cannot do the above with $z_1, \dots z_n$ poles in D.

Defn: (Path) A path $\varphi : [0,1] \to \Omega \subseteq \mathbb{C}$ is a continuous function. This path γ is said to lie in the region Ω.

If $\varphi'(t)$ for $t \in (0,1)$ exist and

$$\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

and if $\lim_{h\to 0^+} \frac{\gamma(0+h) - \gamma(0)}{h}$ & $\lim_{h\to 0^-} \frac{\gamma(1+h) - \gamma(1)}{h}$ exist then the path φ is said to be differentiable. (if $\gamma^{(k)}$ exist $\forall k \Rightarrow$ smooth)

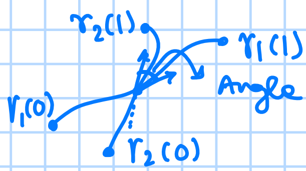
$\varphi : [0,1] \longmapsto$ 

Here $\{ z \in \mathbb{C} \mid z \in \gamma(t) \mid 0 \leq t \leq 1 \}$ is called the trace of φ.

so it is possible trace of path is non-differentiable, but φ is still differentiable

Theorm: Holomorphic functions are conformal map

Defn : Say we have γ_1, γ_2 as two differentiable functions, s.t $\exists\, t_1, t_2 \in (0,1)$ where $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, then we define the angle b/w the curve γ_1 & γ_2 (order is needed) is defind as
$\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$ in case $\gamma_i'(t_i) \neq 0$



The reason of our defination like this is if $\varphi:[0,1]\to\Omega$ is a diff path and say $t_0\in(0,1)$, s.t
$\gamma'(t_0)\neq 0$, then we say the tangent line at z_0, tangent to the curve γ is parallel to "vector" $\gamma'(t_0)$ (complex no)

Recall if $z\in\mathbb{C}\setminus\{0\}$
then $\log(z)=\log|z|+\arg(z)$ — z is treated a a vector in (x,y) plane

$-\pi<\arg(z)<\pi$ → between $-\pi$ and π

Defn: (conformal map) say $\Omega\subseteq\mathbb{C}$ open, then $f:\Omega\to\mathbb{C}$ is called a conformal map if

① f preserves agles between two paths in Ω.

② $\lim_{z\to a}\left|\frac{f(z)-f(a)}{z-a}\right|$ exist $\forall a\in\Omega$

Theorm: If is holomorphic then it is conformal when it is welldefined $(f'(z_0)\neq 0)$

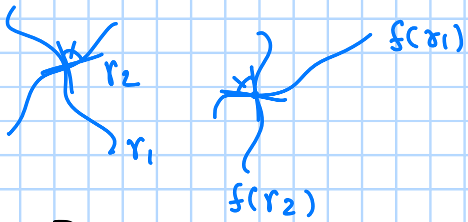
proof: as f is holomorphic
⇒ ② is satisfied

to show ① we have to show that If γ_1,γ_2 are two diff path, $\gamma_1(t_1)=\gamma_2(t_2)=z_0$ and $\gamma_1'(t_1)\neq 0\neq\gamma_2'(t_2)$ then

$$\arg\left(\frac{d}{dt}[f(\gamma_2)](t_2)\right)-\arg\left(\frac{d}{dt}[f(\gamma_1)](t_1)\right) \mod 2\pi \quad ⊛$$

$$=\arg(\gamma_2'(t_2))-\arg(\gamma_1'(t_1)) \mod 2\pi$$

when $\frac{d}{dt}[f(\gamma)](t)\neq 0$



Here, $\frac{d}{dt}[f(\gamma(t))](t)=f'(\gamma(t)).\gamma'(t)$

as when $\gamma'(t)\neq 0$

$$\lim_{h\to 0}\frac{f(\gamma(t+h))-f(\gamma(t))}{h}$$

(γ is diff ⇒ γ is cont)

$$=\lim_{h\to 0}\frac{f(\gamma(t+h))-f(\gamma(t))}{\gamma(t+h)-\gamma(t)}\times\frac{\gamma(t+h)-\gamma(t)}{h}$$

$$=f'(\gamma(t))\times\gamma'(t)$$

similar proof if $\gamma'(t)=0$

now, left side of ⊛ becomes:

$$\arg[f'(\gamma_2(t_2))\gamma_2'(t_2)]$$
$$-\arg[f'(\gamma_1(t_1))\gamma_1'(t_1)]$$
$(\because f$ is well defined$)$

also as $\arg(z_1\cdot z_2)=\arg(z_1)+\arg(z_2) \mod 2\pi$

$\Rightarrow \arg[f'(\gamma_2(t_2))]+\arg(\gamma_2'(t_2))-\arg(f'(\gamma_1(t_1)))-\arg(\gamma_1'(t_1)) \mod 2\pi$

as $\arg(f'(\gamma_2(t_2)))$
$= \arg(f'(z_0))$ $\quad$ $\left(\because f \text{ is well defined } f'(z_0) \neq 0\right)$
$= \arg(f'(\gamma_1(t_1)))$

$\Rightarrow$ left side becomes

$$\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1)) \mod 2\pi$$

So far if $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$ and $\gamma_1'(t_1) \neq \gamma_2'(t_2) + \pi \mod 2\pi$ then f is conformal (trivial case)

in case γ_1 & γ_2 are s.t
$\gamma_1'(t_1) = \gamma_2'(t_2) + \pi \mod 2\pi$
then the angle b/w them is anyways well defined mod 2π.

Defn: (Rectifiable path) A path γ is called rectifiable if γ is of bounded variation
i.e for $\forall$ partition $P = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ of $[0,1]$ we have
$\exists$ $M > 0$ s.t
$$\operatorname{var}(\varphi) = \sup_{\text{partition } P} \left\{ \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \right\} \leq M$$
(Rectifiable paths are B.V)
$\rightarrow$ sup partition

If $\gamma: [0,1] \longrightarrow \mathbb{C}$ is differentiable and has finite length, i.e $\int_0^1 |\gamma'(t)|\,dt < \infty$

then the $\operatorname{var}(\gamma) = \int_0^1 |\gamma'(t)|\,dt$ (diff path, finite $\Rightarrow$ B.V)

Here showing $\operatorname{var}(\gamma) \leq \int_0^1 |\gamma'(t)|\,dt$ is doable (normal Riemann)

to show $\operatorname{var}(\gamma) \geq \int_0^1 |\gamma'(t)|\,dt$ needs the defination of riemann integral
$\left(\int_0^1 |\gamma'(t)|\,dt = \sup_P L(P, \gamma'(t)) = \inf_P U(P, \gamma'(t)) \right)$
$\operatorname{var}(\gamma) \nearrow$ $\quad$ $\searrow \operatorname{var}(\gamma)$

path vs curve vs trace :

Defn: say γ_1, γ_2 are two rectifiable paths, we say that γ_1 is equivalent to γ_2 (denoted by $\gamma_1 \sim \gamma_2$) if $\exists$ (B.V path)
$\varphi\ [0,1] \longrightarrow [0,1]$ strictly increasing s.t $\gamma_2 = \gamma_1 \circ \varphi$
and continious.
(equivalent $\gamma_1 \sim \gamma_2$)

Result: If f is cont, then
$$\int_\gamma f = \int_{\gamma \circ \varphi} f$$
(we are just going to use this result)

any rectifiable curve
φ is cont and strictly inc

we call this a change of variables.

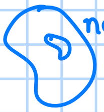
Defn: (Curve) Equivalence classes of all path $\quad$ ($\gamma_1 \sim \gamma_2 \rightarrow$ as they were equivalent)

Note: In change of variables we don't need φ to be differentiable as
$$\left| \int_\gamma f - \sum f(\gamma(t_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

10th march :

Recap: If $\Omega \subseteq \mathbb{C}$ is a convex open set and $f:\Omega \to \mathbb{C}$ is holomorphic then $\exists F:\Omega \to \mathbb{C}$ s.t $F'(z) = f(z)$ (Ω is convex open)

we want to genelise Ω from convex set to any "simply connected open set"



The technique we will use is rigorously define path (v/s curve & trace of path) and then define the notion of homotopy of paths.

now Path $\gamma:[0,1] \to \Omega \subseteq \mathbb{C}$ continious funtion

we consider: Rectifiable path (γ is B.V)

In case γ is piecewise differentiable

$\exists\ 0 < t_1 < \cdots < t_{n-1} < 1 = t_n$, $t_0 = 0$ s.t

$\gamma \in C^1((t_j, t_{j+1}))\ \forall j = 0, 1, \ldots n-1$

& left hand derivative and right hand derivative at t_{j+1} exist & continious.

then $\mathrm{var}(\gamma) = \int_0^1 |\gamma'(t)|\,dt$ (when γ is rectifiable and piecewise smooth)

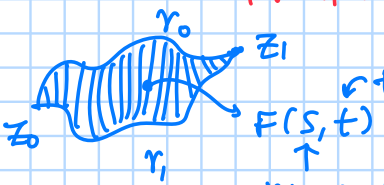
In practice we deal with only piecewise smooth curves.

Defn: (Homotopy of paths) say $\Omega \underset{\text{open}}{\subseteq} \mathbb{C}$ and $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are two paths s.t $\gamma_0(0) = \gamma_1(0)$ & $\gamma_0(1) = \gamma_1(1)$ or they have common end point. Then γ_0 is said to be homotopic to γ_1 in Ω if:
(same points at 0, 1)

$\exists F : [0,1] \times [0,1] \to \Omega$
s.t

① F is jointly continious ($\forall U \underset{\text{open}}{\subseteq} \Omega$, $F^{-1}(U)$ open in $[0,1] \times [0,1]$)

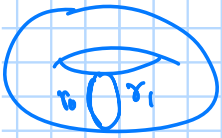
② $F(0,t) = \gamma_0(t)$
& $F(1,t) = \gamma_1(t)$ $\forall t \in [0,1]$



Note: we say $\gamma_0 \underset{h}{\sim} \gamma_1$ if they are homotopic

3-D homotopic paths: (Extra)

$S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\} \subseteq \mathbb{R}^3$



$\gamma_0 \gamma_1^{-1}$ is like a bagle

Not possible on a donut / torus

So look : ① Fundamental group
② Pointcare's conjecture

Theorm: If $f: \underset{\text{open}}{\Omega} \to \mathbb{C}$ is holomorphic then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

(here Ω is any open set)

when $\gamma_0 \underset{h}{\simeq} \gamma_1$
and γ_0, γ_1 are rectifiable.

(for Ω being convex open is true)

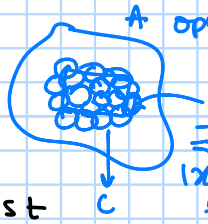
proof: $\gamma_0 \underset{h}{\simeq} \gamma_1$ means that $\exists\, F: [0,1]\times[0,1] \longrightarrow \Omega$ s.t $F(0,t) = \gamma_0(t)$
$F(1,t) = \gamma_1(t)$
$\forall\ 0 \le t \le 1$

also as $[0,1]\times[0,1]$ is compact
$\Rightarrow F([0,1]\times[0,1])$ is compact (∵ F is cont)
$\Rightarrow K \subseteq \Omega$ is compact

now $\underset{\text{compact}}{K} \subseteq \overset{\text{open}}{\Omega}$, $\exists\, \varepsilon > 0$ s.t

$$\bigcup_{z\in K} D(z, 3\varepsilon) \subseteq \Omega$$

when $D(z_0, r) = \{z \mid |z - z_0| < r\}$



$\exists\, \varepsilon$ s.t $|x-y| > \varepsilon$ $\forall x \in C,\ y \in A$

now this is true as if not then $\forall\, l \ge 1$, $\exists\, z_l, w_l$ s.t

$|z_l - w_l| < 1/l$, $z_l \in K$ (this is a contradiction statement)

$w_l \in \mathbb{C}\setminus\Omega$

$\{z_l\} \subseteq K$, compact $\Rightarrow$
$\exists$ a conver subsequence
$\{z_{l_n}\} \subseteq K$ say

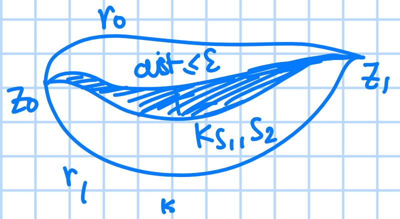
$z_{l_n} \to z \in K$

but $w_{l_n} \to z \in \mathbb{C}\setminus\Omega$ since $\mathbb{C}\setminus\Omega$ is closed (limit point in $\mathbb{C}\setminus\Omega$)

but then $z \in K$
& $z \in \mathbb{C}\setminus\Omega$
this is a contradiction
So $\exists$ such ε.

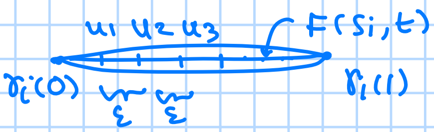
now F is uniformly cont, then given $\varepsilon > 0$
$\exists\, \delta > 0$ s.t

$$|s_1 - s_2| < \delta \Rightarrow \sup_{t\in[0,1]} |F(s_1,t) - F(s_2,t)| < \varepsilon$$



work with $K_{s_1,s_2} = F([s_1,s_2]\times[0,1])$

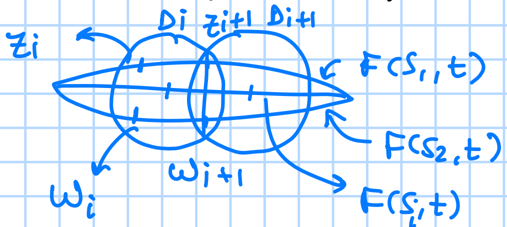
now we construct a nice open cover of $\overset{\text{compact}}{K_{s_1,s_2}}$

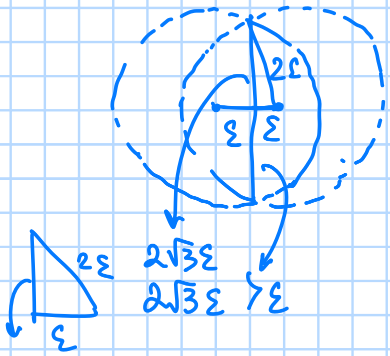


$s_1 \le s_i \le s_2$

$D_0 = D(\gamma_i(0), 2\varepsilon)$
$D_1 = D(u_1, 2\varepsilon)$
$D_2 = D(u_2, 2\varepsilon)$
$\vdots$
$D_K = D(\gamma_i(1), 2\varepsilon)$

then $\quad \operatorname{int}(D_i) \cap \operatorname{int}(D_{i+1}) \neq \phi$
and $\exists\, z_i, z_{i+1}, w_i, w_{i+1} \in \operatorname{int}(D_i)$ s.t





$\leftarrow$ min dist b/w 2 points of intersection

as $2\sqrt{3}\varepsilon > \varepsilon$

$x = \sqrt{3}\,\varepsilon$

as f is hol in Ω, then f is hol in each D_i
$\Rightarrow f$ is hol on $\operatorname{int}(D_i)$ & $\operatorname{int}(D_{i+1})$

$\Rightarrow \exists\, F_i : \operatorname{int}(D_i) \longrightarrow \mathbb{C}$
$F_{i+1} : \operatorname{int}(D_{i+1}) \longrightarrow \mathbb{C}$
s.t $\quad F_i' = f$ on D_i°
$F_{i+1}' = f$ on D_{i+1}°

$\Rightarrow \quad F_{i+1} - F_i = c_i \in \mathbb{C}$
on $D_i^\circ \cap D_{i+1}^\circ$

$\Rightarrow \quad F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1})$
$\Rightarrow \quad F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1})$

now $\displaystyle \int_{z \in F(s_1, *)} f(z) - \int_{z \in F(s_2, *)} f(z)\, dz$

$$= \sum \int_{z_i}^{z_{i+1}} f(z) - \sum \int_{w_i}^{w_{i+1}} f(z)$$

$$= \sum_{i=0}^{N} \Big[F_i(z_{i+1}) - F_i(z_i) \Big] - \sum_{i=0}^{N} \Big[F_i(w_{i+1}) - F_i(w_i) \Big]$$

$$= \sum_{i=0}^{N} \big(F_i(z_{i+1}) - F_i(w_{i+1}) \big) - \sum_{i=0}^{N} \big(F_i(z_i) - F_i(w_i) \big)$$

$=$ $\Rightarrow$ telescopic sum

$$= (F_n(z_{n+1}) - F_n(w_{n+1})) - (F_0(z_0) - F_0(w_0))$$

$$= 0 - 0 = 0$$

as $w_{n+1} = z_{n+1}$
$w_0 = z_0$

$$\Rightarrow \int_{F(s_1,*)} f(z)\,dz = \int_{F(s_2,*)} f(z)\,dz$$

$\Rightarrow$ for $0 < s_1 < \cdots < s_n = 1$
$|s_i - s_{i+1}| < \delta$

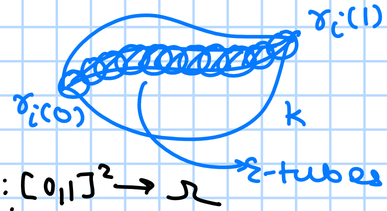
$$\Rightarrow \int_{F(s_0,*)} f = \int_{F(s_1,*)} f = \cdots = \int_{F(s_n,*)} f$$

$$\Rightarrow \int_{\gamma_0} f = \int_{\gamma_1} f$$

11th march :

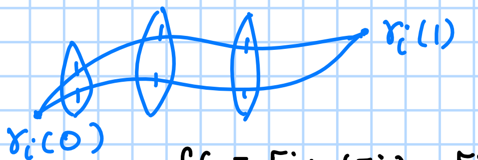
Recap : If $f:\Omega \to \mathbb{C}$ is hol and $\varphi_0, \varphi_1 : [0,1] \to \Omega$ are homotopic $(\varphi_0 \underset{h}{\sim} \varphi_1)$ then

$$\int_{\varphi_0} f = \int_{\varphi_1} f$$



$\exists\, F(s,t) : [0,1]^2 \to \Omega$
① F is jointly cont
② $F(0,t) = \varphi_0(t)$
$F(1,t) = \varphi_1(t)$

say $s_1, s_2 \in [0,1]$, $\exists \delta$ s.t $\forall \varepsilon > 0$
$|s_2 - s_1| < \delta$ then
$|F(s_1,t) - F(s_2,t)| < \varepsilon \ \forall t \in [0,1]$
$\Rightarrow \sup|F(s_1,t) - F(s_2,t)| < \varepsilon$



$$\int_{s_1,i} f = F_{i+1}(z_{i+1}) - F_i(z_i)$$
$$\int_{s_2,i} f = F_{i+1}(w_{i+1}) - F_i(w_i)$$

as f is hol on each disk
$\Rightarrow$ gives primitive on each other
and primite agree on overlap

now, if φ_0, φ_1 are rectifiable ie $\mathrm{var}(\varphi_0), \mathrm{var}(\varphi_1) < \infty$ then
$\mathrm{var}(F(s,*)) < \infty \ \forall\ 0 \le s \le 1$
as
$\mathrm{var}(F(s,*)) = \mathrm{var}(\varphi_0) = \mathrm{var}(\varphi_1)$ as $\varphi_0 \underset{h}{\sim} \varphi_1$

$$\int_{F(s,*)} f = \int_{\varphi_0} f = \int_{\varphi_1} f$$

Defn : (Simply connected domain) $\Omega \subseteq \mathbb{C}$ is called simply connected if $\forall$ path $\gamma_0, \gamma_1 : [0,1] \to \Omega$ with same end points we have $\gamma_0 \underset{h}{\sim} \gamma_1$ in Ω.

Note : Any connected convex set is S.C
as $\varphi_0, \varphi_1 : [0,1] \to \Omega$

$$F(s,t) = \underbrace{(1-s)\gamma_0(t) + s\gamma_1(t)}_{\text{conv combination}} \in \Omega$$

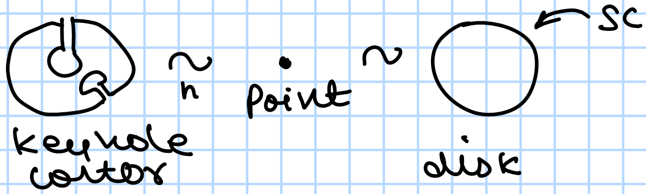
$F(0,t) = \varphi_0(t)$
$F(1,t) = \varphi_1(t)$
and if $\mathrm{var}(\varphi_0) < \infty$, $\mathrm{var}(\varphi_1) < \infty$
$\Rightarrow \mathrm{Var}(F(s,t)) < \infty$
as $F : [0,1] \times [0,1] \to \mathbb{C}$ is a conv combination
$\Rightarrow F([0,1] \times [0,1]) \subseteq \Omega$

Lemma: keyhole contour are also s.c

For this we will need homotopy of topological spaces
If X, Y are two topological spaces, then $X \underset{h}{\simeq} Y$ if

$\exists\, g : X \to Y$
$\quad h : Y \to X$ s.t $g \circ h \underset{h}{\simeq} id_Y$ (in Y)
$\quad$ & $h \circ g \underset{h}{\simeq} id_X$ (in X)
both g, h are continious

If $X \underset{h}{\simeq} Y$ and Y is s.c then X is s.c

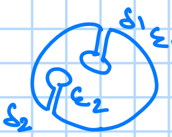


so for now assume that all (multiple) keyhole is s.c

Residue theorm: If $f : \underset{\text{open}}{\Omega} \to \mathbb{C}$ is holomorphic except for a point $z_0 \in D^\circ$
$D \subseteq \Omega$ (Disk) then
$$\frac{1}{2\pi i} \int_{\delta D} f(z)\,dz = res(f, z_0)$$

now, if multiple keyholes are s.t then

$f : \underset{\text{open}}{\Omega} \to \mathbb{C}$ is hol except $z_1, \dots, z_n \subseteq D^\circ$ then, $D \subseteq \Omega$
$$\frac{1}{2\pi i} \int_{\delta D} f(z)\,dz = \sum_{k=1}^{n} res(f, z_k)$$



modulo the existance of multiple keyhole contours

Defn: (Priniple part) f has an isolated singularity at z_0, s.t it is a pole, then
$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k \text{ in a small nbd } U \; (z_0 \in U)$$
Priciple part is $\displaystyle\sum_{k=-n}^{-1} a_k (z - z_0)^k$

Exe: Show that $\gamma_1(t) = e^{2\pi i t}$, $0 \leq t \leq 1$, $\gamma_2(t) = e^{2\pi i (2t)}$, $0 \leq t \leq 1$

do not define same curve, i.e no φ strictly inc s.t $\gamma_1 = \gamma_2 \circ \varphi$

now $\varphi_1 \sim \varphi_2$ (same curve) only if $\exists\, \varphi : [0,1] \to [0,1]$ strictly inc s.t $\varphi_2 \circ \varphi = \varphi_1$
now $\varphi_2 \circ \varphi = \varphi_2(\varphi(t)) = e^{2\pi i \varphi(t) \times 2}$
$= e^{2\pi i t}$
$\Rightarrow \varphi(t) \times 2 = t$ for them to be same

$\Rightarrow \varphi(t) = t/2$

but as $\varphi : [0,1] \to [0,1/2]$

no such φ exist from $[0,1] \to [0,1]$

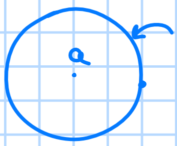
so, $\varphi_1 \not\sim \varphi_2$

proposition: $\varphi : [0,1] \to \Omega$ is a closed curve (i.e $\varphi(0) = \varphi(1)$) then if $a \in \Omega$, $a \notin \{\varphi\}$ then

$$\frac{1}{2\pi i} \int_{\varphi} \frac{1}{z-a} \, dz \text{ is an integer}$$

Defn: (index of closed curve about a) $\frac{1}{2\pi i} \int_{\varphi} \frac{1}{z-a} \, dz$ is called index of closed curve φ about a.

Eg:



unit disk centered at a

$\gamma_n(t) = a + e^{2\pi i n t}$

$t \in [0,1]$

$$\frac{1}{2\pi i} \int_{\varphi} \frac{1}{(z-a)} \, dz = \frac{1}{\cancel{2\pi i}} \int_0^1 e^{-2\cancel{\pi i} nt} \left(e^{\cancel{2\pi i} nt}\right)(\cancel{2\pi i} n) \, dt$$

$$= n \int_0^1 dt = n[t]_0^1 = n$$

proof: we will assume that γ is a smooth curve first (not piecewise smooth)

now define $g : [0,1] \to \mathbb{C}$ by $g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} \, ds$

then $g(0) = 0$

$$g(1) = \int_{\gamma} \frac{1}{z-a} \, dz$$

now by ftc $g'(t) = \dfrac{\gamma'(t)}{\gamma(t) - a}$

now $\frac{\partial}{\partial t} e^{-g(t)}(\gamma(t) - a)$

$$= e^{-g(t)} \gamma'(t) + e^{-g(t)}(-g'(t))(\gamma(t) - a)$$

$$= e^{-g(t)} \gamma'(t) - e^{-g(t)} \gamma'(t)$$

$$= 0$$

$\Rightarrow e^{-g(t)}(\gamma(t) - a) = c$

for $t = 0$

$e^{-g(0)}(\gamma(0) - a) = e^{-g(1)}(\gamma(1) - a)$

$\Rightarrow \gamma(0) - a = e^{-g(1)}(\gamma(1) - a)$

as φ is closed

$\Rightarrow e^{-g(1)} = 1$

$\Rightarrow g(1) = 2\pi i k, \ k \in \mathbb{Z}$

so $g(1) = 2\pi i K = \int_{\varphi} \frac{1}{z-a}\, dz$

$$\Rightarrow K = \frac{1}{2\pi i} \int_{\varphi} \frac{1}{z-a}\, dz \in \mathbb{Z}$$

<u>Theorm</u> : Any holomorphic function $f : \Omega \to \mathbb{C}$ where Ω is s.c open set has a primitive on Ω
so, $\exists F : \Omega \to \mathbb{C}$ s.t $F'(z) = f(z)\ \forall z \in \Omega$

<u>proof</u> : say $z_0 \in \Omega$, and define $F(z) = \int_{\gamma(z_0, z)} f(w)\, dw$

$\hookrightarrow$ any rectifiable path joinig z_0 & z

This is well defined as $\int$ is independent of γ

now, $F(z+h) - F(z) = \int_{h(z, z+h)} f(w)\, dw$

straight line path from z to $z+h$

then we $f(z)$ is cont

$f(w) = f(z) + \Psi(w)$

and then

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h}$$

$$= \lim_{h \to 0} \int_{h(z, z+h)} \frac{f(w)}{h}\, dw$$

$$= \lim_{h\to 0} f(z) + \underbrace{\int \frac{1}{h} \Psi(w)\, dw}_{\to 0 \text{ as } h \to 0}$$

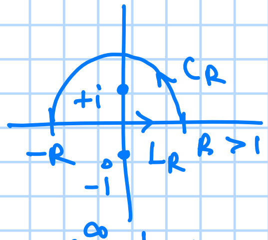
$$= f(z)$$

(Proof of goursats)

<u>cor</u> : $\int_{\varphi} f(w)\, dw = 0$ for any $\varphi : [0,1] \to \Omega$ closed curve φ
(f is hol, Ω is open s.c)

<u>Exe</u> : compute $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ w/o using $\frac{d}{dt}(\tan^{-1}(t)) = \frac{1}{1+t^2}$

now let's take this contor:



let $\gamma_R = L_R + C_R$

now, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} \frac{1}{1+x^2}\, dx$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\, dx \quad (\because \text{absolute convergence})$$

now consider $f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$

$$\frac{1}{2\pi i}\int_{\gamma_R} f(z)\,dz = \mathrm{Res}(f, i) \qquad R>1$$

also, now order $n=1$

so $$\mathrm{res}(f, i) = \lim_{z\to i} \frac{1}{(1-1)!}\left(\frac{d}{dz}\right)^{1-1}(z-i)^1 f(z)$$

$$= \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$$

now $$\left|\int_{C_R} f(z)\,dz\right| \leqslant \pi R \sup_{z\in C_R}|f(z)|$$

$$f(z) = \frac{1}{1+z^2}$$

$$|f(z)| \leqslant \frac{2}{|z|^2}$$

so $$\pi R \sup_{z\in C_R}|f(z)| \leqslant \frac{2}{R^2}$$

so $$\left|\int_{C_R} f(z)\,dz\right| \leqslant \pi R \times \frac{2}{R^2} = \frac{2\pi}{R}$$

as $R\to\infty$

$$\left|\int_{C_R} f(z)\,dz\right| \to 0$$

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R} f(z)\,dz$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{L_R} f(z)\,dz$$

$$+ \lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_R} f(z)\,dz$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{L_R} f(z)\,dz = \frac{1}{2i}$$

$$\Rightarrow \lim_{R\to\infty}\int_{L_R} f(z)\,dz = \pi$$

now $$\int_{L_R} f(z)\,dz = \int_{-R}^{R}\frac{1}{1+x^2}\,dx$$

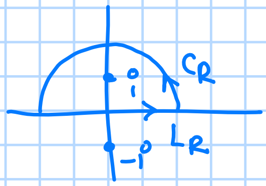
so $$\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{1+x^2}\,dx = \pi$$

12th March :

Today : ① A contour integral
② Removable singularities

Next week : Tuesday and Wednesday → Tutorial
friday → Another proof of Cauchy's residue theorem
(not using keyhole contour) so fill last time gap.

Recap : We proved $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$ (w/o assuming $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$)

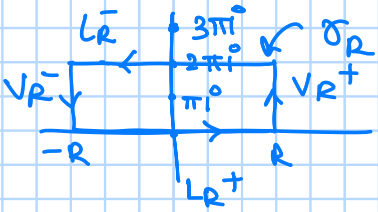


this contour with $f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$ was used

Exe: Find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$, where $0<a<1$

Now let $f(z) = \frac{e^{az}}{1+e^z}$ then for $z=(2n+1)\pi i$
$n\in\mathbb{Z}$ $e^z=-1$
$\Rightarrow f(z)$ not defined

so $f(z) = \frac{e^{az}}{1+e^z}$ $\forall z\in\mathbb{C}\setminus\{(2n+1)\pi i\}_{n\in\mathbb{Z}}$



also $\frac{1}{f(z)} = \frac{1+e^z}{e^{az}}$ is s.t

$= \frac{e^z - e^{\pi i}}{e^{az}} = \dfrac{(z-\pi i) + \frac{(z)^2-(\pi i)^2}{2!} + \frac{(z)^3-(\pi i)^3}{3!} + \cdots}{e^{az}}$

$= \dfrac{(z-\pi i)\left[1 + \frac{(z+\pi i)}{2!} + \cdots\right]}{e^{az}}$

so $h(z)$ and $n=1$

so $\mathrm{Res}(f,\pi i) = \lim_{z\to\pi i}(z-\pi i)f(z)$

also $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \lim_{\substack{R_1\to\infty\\R_2\to\infty}} \int_{-R_2}^{R_1} \frac{e^{ax}}{1+e^x}\,dx$

$= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx$

now by residue theorem,

$\frac{1}{2\pi i}\int_{\gamma_R} f(z)\,dz = \mathrm{Res}(f,\pi i)$

$= \lim_{z\to\pi i} (z-\pi i)f(z)$

$$= \lim_{z \to \pi i} (z - \pi i) \frac{e^{az}}{e^z - e^{\pi i}}$$

$$= \lim_{z \to \pi i} \frac{e^{az}}{\left(\frac{e^z - e^{\pi i}}{z - \pi i}\right)}$$

$$= \lim_{z \to \pi i} \frac{e^{az}}{e^z} \to \text{as exist}$$

and $e^{z_0} = \lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0}$

$\quad\; \| $

$(e^z)'|_{z_0}$

$$= \frac{e^{a\pi i}}{-1}$$

$$\text{Res}(f, \pi i) = -e^{a\pi i}$$

now $\int_{L_R^-} f(z)\,dz \underset{z = t + 2\pi i}{=} \int_R^{-R} \frac{e^{a(t+2\pi i)}}{1 + e^{t+2\pi i}}\,dt = e^{2\pi i a} \int_{+R}^{-R} \frac{e^{at}}{1+e^t}\,dt$

$$= -e^{2\pi i a} \int_{-R}^{+R} f(t)\,dt$$

$$= -e^{2\pi i a} \int_{L_R^+} f(z)\,dz$$

now for V_R^+

$\left| \int_{V_R^+} f(z)\,dz \right| \underset{z = R + it}{\leq} \int_0^{2\pi} \left| \frac{e^{a(R+it)}}{1 + e^{R+it}} \right| dt \leq e^{R(a-1)} \times C$

$\left| \frac{e^{a(R+it)}}{1 + e^{R+it}} \right| = \left| \frac{e^{aR - R}}{e^{-R-it} + 1} \right|$

$= e^{R(a-1)} \times \frac{1}{|e^{-R-it} + 1|}$

some $C > 0$ as $C = \int_0^{2\pi} \left| \frac{1}{e^{-R-it}+1} \right|$

as $a - 1 < 0$ as $R \to \infty$

$$\left| \int_{V_R^+} f(z)\,dz \right| \to 0$$

similarly $\left| \int_{V_R^-} f(z)\,dz \right| \underset{z = -R + it}{\leq} \int_{2\pi}^{0} \left| \frac{e^{a(-R+it)}}{1 + e^{-R+it}} \right| dt$

$\leq C_2 e^{-Ra}$

as $Ra > 0$

as $R \to \infty$

$Ra \to \infty$

and so $C_2 e^{-Ra} \to$

now, $2\pi i(e^{-a\pi i}) = \int_{L_R^-} f + \int_{L_R^+} f + \underbrace{\int_{V_R^+} f}_{\to 0} + \underbrace{\int_{V_R^-} f}_{\to 0}$

$= (-e^{2\pi i a} + 1) \int_{L_R^+} f = (1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{at}}{1 + e^t}\,dt$ when $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t}\,dt = \frac{2\pi i e^{-a\pi i}}{1 - e^{2\pi i a}} = \frac{2\pi i}{e^{a\pi i} - e^{a\pi i}} = \frac{\pi}{\sin(a\pi)}$$

Note: The above exercise can generally be applied to all, just right contour needed.

Theorm: Riemann's result (or theorm) on removable singularities

say $f : \Omega \to \mathbb{C}$ is hol on a nbd U of $z_0 \in \Omega^{\circ}$ with possible singularity at z_0, then the following are equivalent:

① z_0 is a removable singularity of $f(z_0)$

② $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0$

③ $f(z)$ is bdd on $U \setminus \{z_0\}$

proof:

if ① then z_0 is a removable singularity

so $\exists h : U \to \mathbb{C}$ s.t $h(z) = f(z)$ $\forall z \in U \setminus \{z_0\}$ and h is hol

$$\Rightarrow \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) h(z) = 0$$

also $f(z)$ is bounded on $U \setminus \{z_0\}$ is trivial as $h(z)$ is hol $\forall z \in U \Rightarrow h(z)$ is cts $\Rightarrow h(z)$ is bdd

$\Rightarrow f(z)$ is bounded on $U \setminus \{z_0\}$

now supposing ②: $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0$

then it is trivial to see that as $z \to z_0$ $f(z)$ remains bounded

$\Rightarrow f(z)$ is bdd on $U \setminus \{z_0\}$

also to show z_0 is a removable singularity

$$g(z) = \begin{cases} (z - z_0) f(z) & ; z \in U \setminus \{z_0\} \\ 0 & ; z = z_0 \end{cases}$$

then by condition $g(z)$ is cts on U

if we can show $g(z)$ is hol then we are done

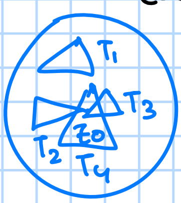
as $g(z_0) = 0$

$g(z) = (z - z_0) h(z)$

$h(z)$ is hol

s.t $f(z) = h(z)$ $\forall z \in U \setminus \{z_0\}$

also we apply morera's theorm by $D(z_0, r) \subseteq U$ then

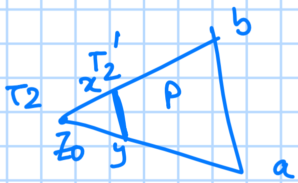


$T_1 \to z_0$ is outside

$T_2 \to z_0$ on vertex

$T_3 \to z_0$ on edge

$T_4 \to z_0$ in interior



now $\int_{T_1} g = 0$ as g is hol when $z \neq z_0$ and $z_0 \notin \operatorname{int} T_1$

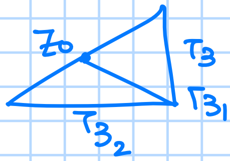
$$\int_{T_2} g = \int_{T_2'} g + \underbrace{\int_P g}_{0}$$

now for $\varepsilon > 0$, x, y can be close enough to z_0

so that $\int_{T_2'} g \leq \varepsilon \times \ln(T_2') \leq \varepsilon \times \ln(T_2)$

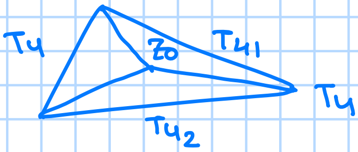
as $\varepsilon \to 0$

$\Rightarrow \int_{T_2'} g = 0$



So $\int_{T_2} g = 0$

$\int_{T_3} g = \int_{T_{3_1}} g + \int_{T_{3_2}} g = 0 + 0$ from previous case



$$\int_{T_4} g = \int_{T_{41}} g + \int_{T_{42}} g + \int_{T_{43}} g + \int_{T_{44}} g$$

$= 0 + 0 + 0 + 0$ from previous case

$= 0$

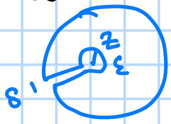
21st march:

Cauchy integral formula:

If $f: \Omega \to \mathbb{C}$ is holomorphic & Ω contains a disk D, then $\forall\, z \in \mathrm{int}(D)$

$$f(z) = \frac{1}{2\pi i} \oint_{\delta D} \frac{f(w)}{w - z}\, dw$$

proof was essentially using keyhole contour



Hole in proof: is showing that f has a primitive in the keyhole $\frac{}{w-z}$ (we cannot use goursats here as not convex)

lemma: If $z \in \mathrm{int}(D(0,1))$ $\quad D(0,1) = \{z \in \mathbb{C} \mid |z| \leq 1\}$

then $\frac{1}{2\pi i} \int_{\delta D(0,1)} \frac{1}{w-z}\, dw = 1$

proof: we parametrise boundary $\quad \delta D(0,1) = \{e^{is} ;\ 0 \leq s \leq 2\pi\}$

Integral $I = \frac{1}{2\pi i} \int_{\delta D(0,1)} \frac{1}{w - z}\, dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\, ds$

$$\varphi(s,t) = \frac{e^{is}}{e^{is} - tz} \qquad \begin{array}{l} 0 \leq s \leq 2\pi \\ 0 \leq t \leq 1 \end{array}$$

$\varphi(s,0) = 1$

$\varphi(s,1) = \frac{e^{is}}{e^{is} - z}$

now let $g(t) = \int_0^{2\pi} \varphi(s,t)\, ds$

we essentially want to use that:

$g(0) = \int_0^{2\pi} \varphi(s,0)\, ds = 2\pi$

and $g(1) = 2\pi \cdot I \quad \left(\text{as } I = \frac{1}{2\pi} \int_0^1 \varphi(s,1) ds \right)$

and now $g(t) = \int_0^{2\pi} \varphi(s,t) dt$

$g(t) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - tz}\, ds$ is continiously differentiable as $\varphi(s,t)$ is cont differentiable as long as denominator not zero

$\Rightarrow e^{is} - tz \neq 0$

and $|e^{is}| = 1$

$|tz| < 1 \ |t| \leq 1$

$\Rightarrow |tz| < 1$ so $e^{is} - tz \neq 0$

so $\varphi(s,t)$ is cont diff

$\Rightarrow \int_0^{2\pi} \varphi(s,t)\, ds$ is cont diff

$$g(t)=\int_0^{2\pi}\varphi(s,t)\,ds$$

$$\Rightarrow \frac{d}{dt}g(t)=\int_0^{2\pi}\frac{d\varphi}{dt}(s,t)\,ds \quad (\because \text{Absolutely conv integral})$$

$$=\int_0^{2\pi}\frac{d}{dt}\left(\frac{e^{is}}{e^{is}-tz}\right)ds$$

$$=\int_0^{2\pi}e^{is}\left[e^{is}-tz\right]^{-2}(-1)(-z)\,ds$$

$$=\int_0^{2\pi}\frac{e^{is}z}{(e^{is}-tz)^2}\,ds$$

If we can find function, $\Phi_t(s)$ s.t

$$\frac{d}{ds}\Phi_t(s)=\frac{e^{is}z}{(e^{is}-tz)^2}$$

then

$$\frac{d}{dt}g(t)=\Phi_t(2\pi)-\Phi_t(0) \quad (\because \text{Fundamental theorm of calculus})$$

$$\Phi_t(s)=\frac{zi}{e^{is}-tz}$$

$$\Rightarrow \Phi_t'(s)=z\cancel{i}(e^{is}-tz)^{-2}\cancel{(-1)}(e^{is})\cancel{(i)}$$

$$=\frac{ze^{is}}{(e^{is}-tz)^2}$$

now $\Phi_t(2\pi)-\Phi_t(0)=\dfrac{ze^{i2\pi}}{(e^{i2\pi}-tz)^2}-\dfrac{ze^{i0}}{(e^{i0}-tz)^2}$

$$=\frac{z}{(1-tz)^2}-\frac{z}{(1-tz)^2}$$

$$\Phi_t(2\pi)-\Phi_t(0)=0$$

$\Rightarrow \dfrac{d}{dt}g(t)=0$

$\Rightarrow g$ is a costant function ($\because g'$ is continious)

$\Rightarrow g(0)=g(1)$

$\Rightarrow 2\pi=2\pi.I$

$\Rightarrow I=1$

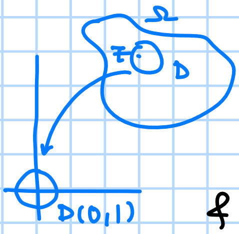
$$\Rightarrow \frac{1}{2\pi i}\int_{\delta D(0,1)}\frac{1}{w-z}\,dw=1$$

Theorm: $f:\Omega\longrightarrow\mathbb{C}$ is hol, $D\subseteq\Omega$ then for $z\in \text{int}(D)$, we have

$$\frac{1}{2\pi i}\int_{\delta D}\frac{f(\omega)}{\omega-z}\,d\omega=f(z)$$

(general case)

proof:



Say the given disk D is $D(a,r) \subseteq \Omega$
now do change of variables to get:

$$\Omega_1 = \left\{ \tfrac{1}{r}(z-a) \,/\, z \in \Omega \right\}$$

& $f(z)$ is now replaced by

$$h(z) = f(a + rz)$$

then the orignal problem becomes

$$\frac{1}{2\pi i} \int_{\delta D(0,1)} \frac{h(w)}{w-z}\, dw = h(z) \rightarrow \text{we want to show this}$$

abuse the notation, write f instead of h (wlog $h = f$, then if we show it, we are done)

prove that $f(z) = \frac{1}{2\pi i} \int_{\delta D(0,1)} \frac{f(w)}{w-z}\, dw$ for $|z|<1$ i.e $z \in \text{int}(D)$

i.e prove $0 = \int_0^{2\pi} \left[\frac{f(e^{is})\, e^{is}}{e^{is} - z} - f(z) \right] ds$ by paramatising $w = e^{is}$

consider function $\varphi(s,t) = \frac{f(z(1-t) + te^{is})\, e^{is}}{e^{is} - z} - f(z)$

now $\varphi(s,0) = \frac{f(z)e^{is}}{e^{is}-z} - f(z)$

$\varphi(s,1) = \frac{f(e^{is})e^{is}}{e^{is}-z} - f(z)$

now let $g(t) = \int_0^{2\pi} \varphi(s,t)\, ds$

also as $|z(1-t) + te^{is}| < 1 \Rightarrow f$ is hol
$\Rightarrow \varphi$ is cont diff in s,t

by FTC g is cont diff

& $g(1) = \int_0^{2\pi} \left[\frac{f(e^{is})e^{is}}{e^{is}-z} - f(z) \right] ds$

$$g(0) = \int_0^{2\pi} \left[\frac{f(z)e^{is}}{e^{is}-z} - f(z) \right] ds$$

$$= f(z) \left[\int_0^{2\pi} \left(\frac{e^{is}}{e^{is}-z} - 1 \right) ds \right]$$

$$= f(z) \int_{\delta D} \frac{dw}{w-z} \frac{1}{i} - 2\pi f(z)$$

$e^{is} = w$
$dw = iw$

$$= f(z) \times \frac{2\pi i}{i} (1) - 2\pi f(z)$$

$= 0$ (from prev lemma)

now $g'(t) = 0 \rightarrow$ then we are done

$$\frac{d}{dt} g(t) = \int_0^{2\pi} \frac{d}{dt} \varphi(s,t)\, ds$$
$$= \int_0^{2\pi} e^{is} f'(z + t(e^{is} - z))\, ds$$

now fix t then

$$\Phi_t'(s) = e^{is} f'(z + t(e^{is} - z))$$

then $\Phi_t(s) = -\frac{i}{t} f(z + t(e^{is} - z))$ for $t \neq 0$ $(t > 0)$

well now $\frac{d}{dt} g(t) = \Phi_t(2\pi) - \Phi_t(0)$
$$= -\frac{i}{t} f(z + t(1-z)) + \frac{i}{t} f(z + t(1-z))$$

$$\frac{d}{dt} g(t) = 0$$

and as g is cont diff $\Rightarrow \left.\frac{d}{dt} g(t)\right|_{t=0} = 0$

so, $\forall t \in [0,1]$ $g'(t) = 0$ and as g is cont
$$\Rightarrow g(t) = C$$
$$\Rightarrow g(0) = g(1) = 0$$
$$\Rightarrow g(0) = \int_0^{2\pi} \frac{f(e^{is}) e^{is}}{e^{is} - z} - f(z)\, ds$$
$$= 0$$
$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{\partial D(0,1)} \frac{f(w)}{w - z}\, dw$$

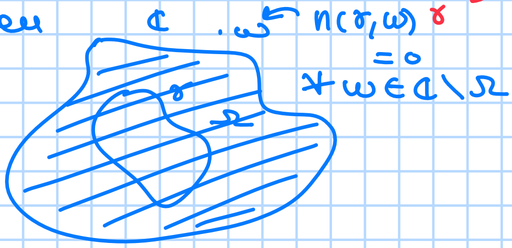
Theorm: (general case of Cauchy's integral formula) let $\Omega \subseteq \mathbb{C}$ open, $f : \Omega \to \mathbb{C}$ is hol, If γ is a closed piecewise smooth

assume in Ω s.t $n(\gamma, w) = 0$ $\forall w \in \mathbb{C} \setminus \Omega$
then for $a \in \Omega - \{\gamma\}$ we have

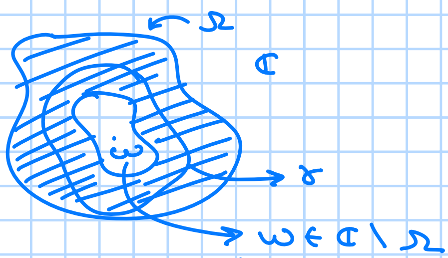
$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{z - a}\, dz$$

where $n(\gamma, w) =$ winding no.
$$= \frac{1}{2\pi i} \int \frac{1}{z - w}\, dz \quad (w \notin \{\gamma\}) \in \mathbb{Z} \text{ (shown)}$$

Eg: Here



but



but $n(\gamma, w) \neq 0$
so not on domain like this

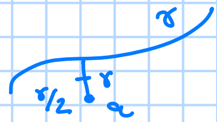
Lemma: (continuity of winding numbers) Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma, a)$ is a continuous function on $\mathbb{C}\setminus\{\gamma\}$. Therefore also $n(\gamma, a)$ is constant on connected component of $\mathbb{C}\setminus\{\gamma\}$

proof: Here

$G = \mathbb{C}\setminus\{\gamma\}$ is open $\Rightarrow$ has countable many connected components

for notation, write $f(a) = n(\gamma, a)$ for $a \in G$

& say $r = d(a, \{\gamma\}) = \inf\{|z-a| \mid z \in \{\gamma\}\}$



If $b \in G$ s.t

$|a-b| < \delta < \frac{r}{2}$, we have

$$|f(a) - f(b)| = \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz\right|$$

$$= \frac{1}{2\pi}\left|\int_\gamma \frac{(z-b)-(z-a)}{(z-a)(z-b)}\, dz\right|$$

$$= \frac{|a-b|}{2\pi}\left|\int_\gamma \frac{1}{(z-a)} \times \frac{1}{(z-b)}\, dz\right|$$

$$\leqslant \frac{|a-b|}{2\pi} \int_\gamma \frac{|dz|}{|z-a|} \times \frac{1}{|z-b|}$$

for $|a-b| < \frac{r}{2} \Rightarrow |z-a| \geqslant r > \frac{r}{2}$

(By definition)

similarly $\Rightarrow |z-b| > \frac{r}{2}$

So $|f(a) - f(b)| < \frac{1}{2\pi} \times \delta \times \frac{2}{r} \times \frac{2}{r} \times \int_\gamma |dz|$

γ is rectifiable or B.V

$$= \frac{1}{2\pi}\, \delta \left(\frac{4}{R^2}\right) \operatorname{var}(\gamma)$$

So as a is fixed $\Rightarrow R$ is fixed and so as $\delta \longrightarrow 0$

$|f(b) - f(a)| \longrightarrow 0$

$\Rightarrow f(b) = f(a)$

given $\varepsilon > 0$, choosing $\delta = \frac{r^2 \pi \varepsilon}{2\operatorname{var}(\gamma)}$

shows: $|a-b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon$

as f is cont and only integer values $\Rightarrow$ const on connected component

25th march:

Theorm: (general case of Cauchy's integral formula) let $\Omega \subseteq \mathbb{C}$ open, $f: \Omega \to \mathbb{C}$ is hol, If γ is a closed piecewise smooth

assume in Ω s.t $n(\gamma, w) = 0 \ \forall w \in \mathbb{C} \setminus \Omega$
then for $a \in \Omega - \{\gamma\}$ we have

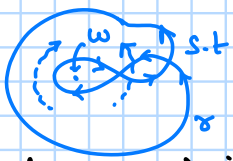
$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{z-a}\, dz$$

where $n(\gamma, w) =$ winding no.
$= \frac{1}{2\pi i} \int_\gamma \frac{1}{z-w}\, dz \ (w \notin \{\gamma\}) \in \mathbb{Z}$ (shown)

proof: Define $\varphi : \Omega \times \Omega \to \mathbb{C}$ by $\varphi(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & ; z \neq w \\ f'(z) & ; z = w \end{cases}$

now φ is cont on $\Omega \times \Omega$ (trivial)
& if w is fixed, then
$\varphi(z, w)$ is holomorphic on Ω as (trivial) function of z & vice-versa

$H = \{ w \mid n(\gamma, w) = 0 \}$



s.t $n(\gamma, w) = 0$

last time we showed that $n(\gamma, w)$ is cont on $\mathbb{C} \setminus \{\gamma\}$ ← open set
& $n(\gamma, w)$ is const on all connected components

$H =$ inverse image $\{ B(0, \frac{1}{2}) / n(\gamma, \cdot) \}$
open in $\mathbb{C} \setminus \{\gamma\}$

$n^{-1}(B(0, \frac{1}{2})) = H$
open as U open $\Rightarrow f^{-1}(U)$ open

& now $H \cup \Omega = \mathbb{C}$ (By hypothesis of the theorm)

now define $g: \mathbb{C} \to \mathbb{C}$

$$g(z) = \begin{cases} \int_\gamma \varphi(z, w)\, dw & ; z \in \Omega \\ \int_\gamma \frac{f(w)}{w-z}\, dw & ; z \in H \end{cases}$$

now let's show g is well defined as for $z \in H \cap \Omega$

then $g(z) = \int_\gamma \varphi(z, w)\, dw = \int_\gamma \frac{f(w) - f(z)}{w-z}\, dw$
$= \int_\gamma \frac{f(w)}{w-z}\, dw - \int \frac{f(z)}{w-z}\, dw$ ← $f(z) n(\gamma, z)$, $z \in H$, $= 0$
$= \int_\gamma \frac{f(w)}{w-z}\, dw$

so $g(z)$ is well defined.
now, g is cont as both $\varphi(z,\omega)$ and $\frac{1}{z-\omega}$ are continius fution of z

Assume: we have proved that $g(z)$ is holomorphic on $\mathbb{C}$ (This assumption will be proved later)

now, $\{\gamma\}$ is a compact set $\Rightarrow \sup_{z\in\{\gamma\}} |f(z)|$ is finite

now $\left|\lim_{z\to\infty} g(z)\right| = \left|\lim_{z\to\infty} \int \frac{f(\omega)}{\omega - z} d\omega\right|$

now, we may replace Ω by bounded set $D(0,R)$ s.t $\{\gamma\} \subseteq D(0,R)$
$\forall z \notin D(0,R)$
we have $z\in H \Rightarrow n(\gamma,z)=0$

so
$$\left|\lim_{z\to\infty} g(z)\right| = \left|\lim_{z\to\infty} \int \frac{f(\omega)}{\omega - z} d\omega\right|$$
$$\leq \lim_{z\to\infty} \int_{\omega\in\gamma} \frac{|f(\omega)|}{|\omega - z|} |d\omega|$$
$$\leq \lim_{z\to\infty} \frac{\sup|f(\omega)| \, \mathrm{var}(\gamma)}{|\omega - z|}$$
$$= 0$$

now if $g : \mathbb{C} \longrightarrow \mathbb{C}$ s.t $\lim_{z\to\infty} g(z) = 0$ thus

given $\varepsilon > 0$, $\exists R > 0$ s.t $|z| > R \Rightarrow |g(z)| < \varepsilon$
& $D(0,R)$ is compact
$\Rightarrow g(z)$ is bounded on $D(0,R)$
$\Rightarrow g(z)$ is bonded on $\mathbb{C}$

$\Rightarrow$ By liouvilles theorm
g is constant $\forall \mathbb{C}$
$\Rightarrow g \equiv 0 \quad \left(\because \lim_{z\to\infty} g(z) = 0\right)$

so $0 = \int_\gamma \frac{f(\omega) - f(z)}{\omega - z} d\omega$ for $z \in \Omega \setminus \{\gamma\}$

$$\Rightarrow 2\pi i \, n(\gamma,z) f(z) = \int_\gamma \frac{f(\omega)}{\omega - z} d\omega$$

so
$$n(\gamma,z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{\omega - z} d\omega$$

<u>lemma</u>: say γ is rectifiable, closed & φ is a fuction which is continious on $\{\gamma\}$ Then $\forall m \geq 1$
$F_m(z) = \int_\gamma \frac{\varphi(\omega)}{(\omega - z)^m} d\omega$ for $z \notin \{\gamma\}$ is holomorphic on $\mathbb{C} \setminus \{\gamma\}$
also $F_m'(z) = m F_{m+1}(z)$

If we take $\varphi = f$, for $z \in H$
$\varphi = f - 1$, for $z \in \Omega$
we show $g(z)$ is hol on $z \in \mathbb{C} \setminus \{\gamma\}$

as g is cont on $\mathbb{C}$

$\Rightarrow$ for $z \in \{\gamma\}$ $\lim_{w \to z} g(w) = g(z)$

then show $z \in \{\gamma\}$ is a removable singularity

as $\lim_{w \to z} (w-z) g(w) = 0$

so g is hol on $\mathbb{C}$

proof: $F_m(z)$ is cont for $z \in \mathbb{C} \setminus \{\gamma\}$

now

$$\lim_{w \to z} F_m(w) = F_m(z)$$

take $z \in \mathbb{C} \setminus \{\gamma\}$, $z \in D$ in a compact component

for $w \in D$

as $w \longrightarrow z$

$$r = \inf\{ |z-u| \mid u \in \{\gamma\} \} > 0$$

take

$$w \in D(z, r/1000)$$

$$\text{then } F_m(w) - F_m(z) = \int_\gamma \frac{\varphi(u)\, du}{(u-w)^m} - \int_\gamma \frac{\varphi(u)}{(u-z)^m} du$$

$$= \int_\gamma \varphi(u) \left[\frac{1}{(u-w)^m} - \frac{1}{(u-z)^m} \right] du$$

$$A^m - B^m = (A-B)\left(\sum_{k=0}^{m-1} A^k B^{m-k-1}\right)$$

$$A - B = \underbrace{\frac{w-z}{(u-w)(u-z)}}_{\text{Bounded}} \to 0 \text{ as } w \to z$$

$$\sum A^k B^{m-k-1} = \sum \underbrace{\left(\frac{1}{u-w}\right)^k}_{\text{Bounded}} \underbrace{\left(\frac{1}{u-w}\right)^{n-k-1}}_{\text{Bounded}}$$

$$\Rightarrow \lim_{w \to z} F_m(w) = F_m(z)$$

$\therefore$ F_m is continious

also as $F_m(z) = \int_\gamma \frac{\varphi(w)\, dw}{(w-z)^m}$

$\Rightarrow F_m(z)$ is holomorphic

$\left(\text{as } F_m(z) = \int f \text{ is cont} \right.$

$\left. \Rightarrow F'_m(z) = f \text{ so hol} \right)$

General cauchy's theorm :

let $\Omega \subseteq \mathbb{C}$ & $f : \Omega \longrightarrow \mathbb{C}$ is hol say $\gamma_1 \dots \gamma_m \subseteq \Omega$ are closed open rectifiable s.t

$$n(\gamma_1, w) + \cdots n(\gamma_m, w) = 0$$

$\forall\ w \in \mathbb{C} \setminus \Omega$, then $\forall\ a \in \Omega \setminus \left\{ \bigcup_{i=1}^m \gamma_i \right\}$

we have $$f(a) \sum_{k=1}^n n(\gamma_k, a) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz$$

Note : Idea to prove is that Replace H by

$$H = \{ w \mid \sum n(\gamma_i, w) = 0 \}$$

rest of proof is like earlier.

Note : one more way to have cauchy's theorm is

$f : \Omega \longrightarrow \mathbb{C}$ is hol

then $$\sum_{k=1} \int_{\gamma_k} f(z)\, dz = 0$$

Theorm: (Cauchy's theorm) If $f: \Omega \to \mathbb{C}$ is hol

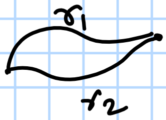
$$\sum_{K=1} \int_{\gamma_K} f(z)\,dz = 0$$

proof: Replace $f(z)$ by $f(z)(z-a)$ to get the result $\left(\because f(a)(a-a)\sum n = \sum_{=0} \int\right)$

Theorm: (one more version of Cauchy's) If $f: \Omega \to \mathbb{C}$ is hol, say γ is rectifiable & closed & $\gamma \underset{h}{\sim} 0$

then $\int_\gamma f = 0$, 0 is a const curve at a point

proof:



$\gamma_1 \underset{h}{\sim} \gamma_2$ if $\exists\, \varphi [0,1] \times [0,1] \to \mathbb{C}$, continious s.t

$\varphi(0,t) = \gamma_0(t)$
$\varphi(1,t) = \gamma_1(t)$
$\varphi(s,0) = \gamma_0(0) = \gamma_1(0)$
$\varphi(s,1) = \gamma_0(1) = \gamma_1(1)$

$\gamma \underset{h}{\sim} 0$ means that $\exists\, a \in \{\gamma\}$ $(a = \varphi(0))$ s.t

$\exists\, \varphi [0,1] \times [0,1] \to \mathbb{C}$ cont s.t

$\varphi(0,t) = \varphi(t)$
$\varphi(1,t) = a = \varphi(0)$

$\varphi(s,0) = \varphi(s,1) = a = \gamma(0)$

now if $\gamma \underset{h}{\sim} 0 \Rightarrow n(\gamma, \omega) = 0 \ \forall\, \omega \in \mathbb{C} \setminus \Omega$
(we will assume this and not cover it) (as  everything outside 0)

Residue theorm:

Did this using keyhole contours

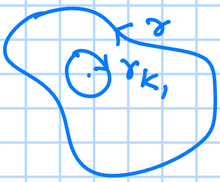
Theorm: now $\Omega \subseteq_{\text{open}} \mathbb{C}$ $f: \Omega \to \mathbb{C}$ is hol except $a_1 \ldots a_n \in \Omega$, say γ is closed rectifiable & $\gamma \underset{h}{\sim} 0$ $\subseteq \Omega \setminus \{a_1 \ldots a_n\}$

then $$\frac{1}{2\pi i} \int_\gamma f = \sum_{K=1}^{n} n(\gamma, a_K)\, \mathrm{Res}(f, a_K)$$

Proof: let $m_K = n(\gamma, a_K)$

choose small disk $D(a_K, r_K)$ s.t r_K is small enough so $D(a_K, r_K) \subseteq \Omega \setminus \{\gamma\}$ $\forall\ 1 \le K \le n$

Then Paramatrise $\partial D(a_K, r_K)$ $\gamma_K(t) = a_K + r_K e^{-2\pi i m_K t}$



$$\Rightarrow n(\gamma, a_j) + \sum_{K=1}^{m} n(\gamma_K, a_j) = 0 \quad \forall\ 1 \le j \le m$$

$= m_j - m_j = 0$
$\left(\because n(\gamma, a_j) = m_j,\ \sum n(\gamma_K, a_j) = -m_j\right)$

and $n(\gamma, a) + \sum_{k=1}^{m} n(\gamma_k, a) = 0 \quad \forall\, a \notin \Omega \setminus \{a_1, \dots a_n\}$

$\Rightarrow n(\gamma, a) + \sum n(\gamma_k, a) = 0 \;\forall\; a \in \mathbb{C} \setminus (\Omega \setminus \{a_1, \dots a_n\})$

$\Rightarrow 0 = \int_\gamma f + \underbrace{\sum_{k=1}^{m} \int_{\gamma_k} f}$

$= -\sum_{k=1}^{m} n(\gamma, a_k)\, \mathrm{Res}(f, a_k)\,(2\pi i)$ ← By normal Residue formula

$\Rightarrow \frac{1}{2\pi i} \int_\gamma f = \sum_{k=1}^{m} n(\gamma, a_k)\, \mathrm{Res}(f, a_k)$

28th march:

Today: ① understanding the nature of singularity
② laurent series expansion

Recall: Isolated singularities:
f has an iso. at $z=a$ if $\exists\, r>0$ s.t
f is hol on $D(a,r)\setminus\{a\}$

kinds: ① Removable
② pole

(we didn't do a result which tells us it is exhaustive)

Removable: $\lim_{z\to a}(z-a)f(z)=0 \Leftrightarrow z=a$ is removable

(a nice rule to show something is removable $\lim_{z\to a}(z-a)f(z)=0 \Leftrightarrow z=a$ is removable)

Pole: $\tilde{f}=\begin{cases} 1/f & ; z\neq a \\ 0 & ; z=a\end{cases}$ is hol on $D(a,r)$ in which case

$f(z)=\dfrac{h(z)}{(z-a)^m}$ where h is non vanishing on $D(a,r)$ & hol

and $m\geq 1$ pole at $z=a$ of order m

lemma: $z=a$ is a pole of f $\Leftrightarrow \lim_{z\to a}|f(z)|=\infty$

(this is a test for pole $\lim_{z\to a}|f(z)|=\infty$)

proof: $(\Rightarrow)$ If $z=a$ is a pole
$\exists\, h$ s.t
$f(z)=\dfrac{h(z)}{(z-a)^m}$, h is hol at $z=a$
$m\geq 1$

$$\Rightarrow \lim_{z\to a}|f(z)|=\lim_{z\to a}\frac{|h(z)|}{|z-a|^m}$$

$$=\lim_{z\to a}\frac{|h(a)|}{|z-a|^m}=\infty$$

($\because h(a)\neq 0$ as non-vanishing function)

$(\Leftarrow)$ given $\lim_{z\to a}|f(z)|=\infty$ then

f is holomorphic on $D(a,r)\setminus\{a\}$ s.t
$\lim_{z\to a}|f(z)|=\infty$

then $\lim_{z\to a}\dfrac{1}{|f(z)|}=0$

$\Rightarrow \lim_{z\to a}\dfrac{1}{f(z)}=0$

Since $\lim_{z\to a}|f(z)|=\infty$, $\exists\, r_1>0$ s.t
f is hol & non-vanishing in $D(a,r_1)\setminus\{a\}$

now if $g(z)=\begin{cases} 1/f(z) & ; z\neq a \\ 0 & ; z=a\end{cases}$

then $g(z)$ is hol on $D(a,r_1)$

(By definition of poles $\exists\,\tilde{f}$)

taking $g=\tilde{f}\Rightarrow z=a$ is a pole of $f(z)$

If $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ in some disk $D(a,r) \setminus \{a\}$ then $z=a$ is removable
(one more test)

lemma: $z=a$ is a pole of $f \Leftrightarrow \exists r>0$ s.t $f(z) = \sum_{n=-m}^{\infty} a_n (z-a)^n$
for $z \in D(a,r)\setminus\{a\}$, $m \geq 1$

proof: Trivial from defination (one more test for poles)

Essential singularity

Defn: An isolated singularity which is neither removable, not a pole

(Isolated − Rem − Pole = Essen)

Eg: $f(z) = e^{\frac{1}{z}}$ in $D(0,1)\setminus\{0\}$

at $z=0$ neither rem, nor a pole

formaly: $e^{\frac{1}{z}} = \sum_{n\geq 0} \frac{1}{n!\, z^n} = \sum_{n \leq 0} \frac{z^n}{(-n)!}$ (goes till $-\infty$ to ∞)

Now, lets consider $|e^{1/z}|$ for $z = x+iy$

$\frac{1}{z} = \frac{\bar{z}}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$ $(z \neq 0)$

then $|e^{1/z}| = |e^{x/x^2+y^2}|\, \underbrace{|e^{-iy/x^2+y^2}|}_{=1}$

$= e^{x/x^2+y^2}$

let $Re(z)=0$
$Im(z) \to 0$

then $\lim_{z\to 0} |e^{1/z}| = \lim_{z\to 0} 1 = 1$ Not pole from this argument

$\left(\because \lim_{z\to a} |e^{z}| = \infty \text{ for pole} \right)$

also $Im(z) = 0$
$Re(z) \to 0$

$\lim_{z\to 0} |e^{1/z}| = \lim_{x \to 0} e^{1/x}$

for $x \to 0^+$ $\lim e^{1/x} = \infty$ Not removable
(as $f(z) = h(z)$ then $h(z)$ cannot go to infitiy)

$x \to 0^-$ $\lim e^{1/x} = 0$

so this cannot be a pole

Tutorial: given $R \geq 0$ find $z \to 0$ s.t $\lim_{z\to 0} |e^{1/z}| = R$ along a path
(tutorial question this week)

Theorm: (Casorati-Weierstrass theorm) If $z=a$ is an essential singularity of $f(z)$, then given any small $r>0$ we have

$f(D(a,r)\setminus\{a\})$ is dense in $\mathbb{C}$

proof: lets assume this does not happen, then $\exists\, w \in \mathbb{C}$ & $\exists\, \delta > 0$ s.t
$|f(z) - w| > \delta \quad \forall\, z \in D(a,r)\setminus\{a\}$

$\Rightarrow$ consider $g(z) = \frac{1}{f(z)-w}$ in $D(a,r)\setminus\{a\}$

($f(D(a,r)\setminus\{a\})$ is dense in $\mathbb{C}$ $\Leftrightarrow$ given any $z_0 \in \mathbb{C}$ & $r_2 > 0$ $\exists\, w \in D(a,r)\setminus\{a\}$ s.t $f(w) \in D(z_0, r_2)$)

now, $g(z)$ is hol on $D(a,r)\setminus\{r\}$ as $f(z) - w \neq 0$ & $f(z)$ is hol now, $\lim_{z\to a} |(z-a)\, g(z)| \leq \lim_{z\to a} |z-a| \frac{1}{\delta} = 0$

$\Rightarrow z=a$ is a removable singularity of $g(z)$

if $g(a)=0$

i.e $\lim_{z\to a} \frac{1}{f(z)-w} = 0$

$\Rightarrow \lim_{z\to a} |f(z)-w| \to \infty$

$\Rightarrow \lim_{z\to a} |f(z)| \to \infty$

$\Rightarrow z=a$ is a pole

else $\lim_{z\to a} g(z) \neq 0 \Rightarrow z=a$ is a removable singularity of $f(z)$

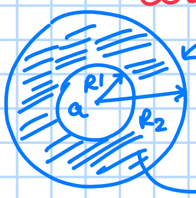
as $f(z)$ does not have pole or removable at $z=a$ this is a contradiction (contradiction being it will have a pole or removable sing)

laurent series expansion:

Defn: Annulus of radii $R_1 < R_2$ as $Ann(a; R_1, R_2) = \{z \mid R_1 \leq |z-a| \leq R_2\}$ and $ann(a; R_1, R_2) = int(Ann(a; R_1, R_2))$

Defn: (laurent series expansion) let f be holomorphic on $ann(a; R_1, R_2)$ then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ for $z \in ann(a; R_1, R_2)$. This series converges absolutely and uniformly on $Ann(a; \gamma_1, \gamma_2)$ where $R_1 < \gamma_1 < \gamma_2 < R_2$ and $a_n = \frac{1}{2\pi i} \int_{\delta D(a,\gamma)} \frac{f(z)}{(z-a)^{n+1}} dz$

$\gamma_1 \leq \gamma \leq \gamma_2$



$Ann(a; R_1, R_2)$

f is hol on $ann(a; R_1, R_2)$

To show this result we can write $f(z) = f_1(z) + f_2(z)$ where $f_2(z)$ is hol on $D(a, \gamma_2)$ & f_1 is hol on $int(D(a, R_2) \setminus D(a, R_1))$

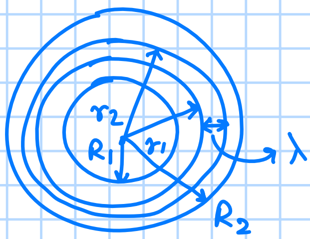
and then use contour integration and lemma done previously for $F_m(z)$

use the below lemma

lemma: If γ is rectifiable path and φ is a funtion which is cont on γ then $\forall m \geq 1$

$$F_m(z) = \int_\gamma \frac{\varphi(w)}{(w-z)^m} dw$$

is hol on $\mathbb{C} \setminus \{\gamma\}$ and $F'_m = m F_{m-1}$



for $R_1 < \gamma_1 < \gamma_2 < R_2$ we have

$$\int_{\gamma_1} f = \int_{\gamma_2} f \quad \text{as} \quad \gamma_1 \underset{h}{\sim} \gamma_2$$

$\gamma_1 = \delta D(a, \gamma_1)$
$\gamma_2 = \delta D(a, \gamma_2)$

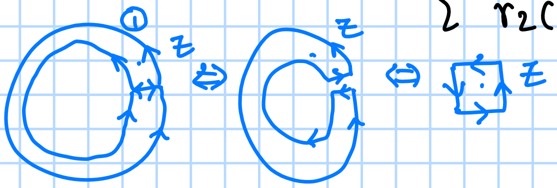
and so for $\gamma_1 \leq \gamma \leq \gamma_2$ same value of integral

given $z \in ann(a; R_1, R_2)$, then $\exists R_1 < \gamma_1 < \gamma_2 < R_2$ and $z \in ann(a; \gamma_1, \gamma_2)$ and consider the concatenation of path

$\gamma = \gamma_2 \ominus \lambda \ominus \gamma_1 \oplus \lambda$

where concatenation of paths is if 2 paths s.t. $\gamma_1(1) = \gamma_2(0)$

$$\gamma = \gamma_1 \oplus \gamma_2 = \begin{cases} \gamma_1(2t) & ; 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & ; 1/2 \leq t \leq 1 \end{cases}$$



so notice $\gamma \underset{h}{\sim} 0$ inside $\mathrm{ann}(a; r_1, r_2)$

So by cauchy's theorm:

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\omega)}{\omega - z}\, d\omega - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\omega)}{\omega - z}\, d\omega$$

or we can also use

$$\sum_{\gamma_i} n(\gamma_i, a) f(a) = \frac{1}{2\pi i}\sum \int_{\gamma_i} \frac{f(\omega)}{\omega - z}\, d\omega$$

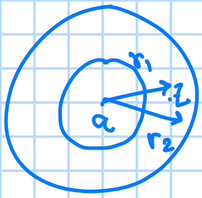
now, let $f_2(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\omega)}{\omega - z}\, d\omega$ then by lemma

f_2 is hol on $\mathrm{int}(D(a, r_2))$

1^{st} Apr :

Laurent series expansion :

f is hol on $\operatorname{ann}(a; R_1, R_2)$



now $f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\omega)}{\omega - z} d\omega - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} d\omega$

and then we can solve

$\Omega = \operatorname{ann}(a; R_1, R_2)$

Another argument : If Ω is s.t $\forall u \in \mathbb{C} \setminus \Omega$ we have

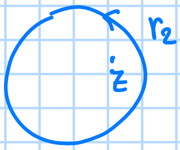
$$\sum_i n(\gamma_i, u) = 0 \quad \forall z \in \Omega$$

$$\sum_i n(\gamma_i; z) f(z) = \frac{1}{2\pi i} \sum \int_{\gamma_i} \frac{f(\omega)}{\omega - z} d\omega$$

replace γ_1 to $-\gamma_1$

$$f(z) = \underbrace{\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\omega)}{\omega - z} d\omega}_{f_2(z)} - \underbrace{\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} d\omega}_{f_1(z)}$$

using lemma we get $f_2(z)$ is hol on $\mathbb{C} \setminus \{\gamma_2\}$ (already proved)



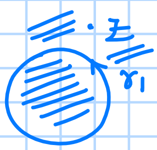
$$\Rightarrow f_2(z) = \sum_{n=0}^{\infty} \frac{f_2^{(n)}(a)}{n!} (z-a)^n$$

& $f_2^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega$ (cauchy)

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{(as holomorphic)}$$

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega$$

now, $f_1(z) = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} d\omega$ is hol on $\mathbb{C} \setminus \{\gamma_1\}$



let us map $\mathbb{C} \setminus D(a, r_1) \longrightarrow \operatorname{int}(D(0, \frac{1}{r_1}))$

$$z \longmapsto \frac{1}{z-a}$$

so, f_1 is hol on $\mathbb{C} \setminus D(a, r_1)$

define $g : \operatorname{int}(D(0, \frac{1}{r_1})) \setminus \{0\} \to \mathbb{C}$

by : $f_1(z) = g\left(\frac{1}{z-a}\right)$ then

$\omega = \frac{1}{z-a}$ then g is hol for $\omega \in \operatorname{int}(D(0, \frac{1}{r_1}))$

small gap, $\omega = 0$ is removable singularity of g

as $w \to 0$ is same as $z \to \infty$

$$\lim_{w\to 0} g(w) = \lim_{z\to\infty} f_1(z) = \lim_{z\to\infty} \int_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

$$\left|\lim_{z\to\infty} f_1(z)\right| \leq \left|\int_{\gamma_1} \frac{f(w)}{w-z}\right| \leq \frac{\sup|f(w)|2\pi r_1}{2|z|} \to 0$$

for big z as $z \to \infty$

$$\Rightarrow \lim_{z\to\infty} f_1(z) = 0 = \lim_{w\to 0} g(w)$$

By expanding $g(w)$ in $\operatorname{int}(D(0,\frac{1}{r}))$

$$g(w) = \sum_{n=0}^{\infty} B_n w^n \quad \& \quad B_0 = 0 \text{ as } g(0)=0$$

Hol on $D(0, 1/r)$

$$\Rightarrow g(w) = f_1(z) = \sum_{n\geq 1}^{\infty} B_n \frac{1}{(z-a)^n}$$

where $B_n = \frac{n!}{2\pi i} \int_{\gamma} \frac{g(w)}{(w)^{n+1}}\,dw$

$\gamma = \delta D(0,s)$, $s < \frac{1}{r_1}$

take $s = \frac{1}{r}$ (same r for expansion of f_2)

$$\Rightarrow B_n = \frac{n!}{2\pi i} \int_{\gamma} \frac{g(w)}{w^{n+1}}\,dw$$

put $u = \frac{1}{z-a}$ $\quad g(w) = f_1(z)$

$$du = \frac{-1}{(z-a)^2}\,dz$$

$$b_n = -\int_{\delta D(a,r)} \frac{(n)!}{2\pi i} \frac{f_1(z)}{(z-a)^{n+1}}\,dz$$

$$= -\frac{(-n)!}{2\pi i} \int_{\delta D(a,r)} \frac{f_1(z)}{(z-a)^{n+1}}\,dz \quad ((-n)! = (n)!)$$

minus sign cancels out $f_1(-) - \infty$

so $f_1(z) = \sum_{n\leq -1} a_n (z-a)^n$

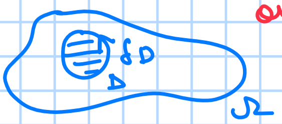
Argument principle:

Recall: f is called meromorphic on Ω (open) if the only singularities of f are either removable or poles.

Eg: $f(z) = \frac{1}{z^2}\sin(z)$ has a pole at $z=0$

Note: In the above $f(z)$, we say f is meromorphic on $\mathbb{C}$ with pole $(m=1)$ at $z=0$

Theorem: (Special case of Arg principle) f is meromorphic on Ω, $D \subseteq \Omega$ $\gamma = \delta D$ & f has no poles/zeros on $\{\gamma\}$ then $\forall z \in \operatorname{int}(D)$



(we can use to find no of zeroes in a region) $\frac{1}{2\pi i}\int_{\delta D} \frac{f'(z)}{f(z)}\,dz = \#$ zeroes of f inside $\mathrm{int}(D)$ $- \#$ poles of f outside $\mathrm{int}(D)$

proof: as $\log(a_1 a_2) = \log(a_1) + \log(a_2) \mod (2\pi)$
in particular

$$\log(f_1(z).f_2(z)) = \log(f_1(z)) + \log(f_2(z)) \mod 2\pi$$

for small nbd $U \ni z$

$$\log(f_1 f_2) = \log f_1 + \log f_2 + c \quad \text{const}$$

diff:

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

more generally:

$$\frac{\left(\prod_{k=1}^{n} f_k\right)'}{\prod_{k=1}^{n} f_k}(z) = \sum_{k=1}^{n} \frac{f_k'(z)}{f_k(z)}$$

for $z_0 \in \mathrm{int}(D)$ f has a zero of order n at z_0
$f(z) = (z-z_0)^n g(z)$
$z \in U \ni z$ small nbd
g is non-van hol on U

then

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1} g(z) + (z-z_0)^n g(z)}{(z-z_0)^n g(z)}$$

$$= \frac{n}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{hol on } U}$$

$\Rightarrow \frac{f'}{f}$ has pole of order $n=1$ (s.pole) for $z=z_0$ with res $= n$

now if f has pole of order m

$$f(z) = (z-z_0)^{-m} h(z)$$

non-van on $U \ni z$

$$\frac{f'(z)}{f(z)} = \frac{-m}{(z-z_0)} + \frac{h'}{h}$$

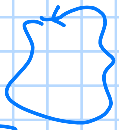
hol on U

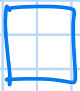
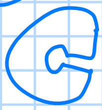
pole of order1 (s.pole)
Res $= -m$

$\Rightarrow \frac{f'}{f}$ is meromorphic on $\mathrm{int}(D)$ and has poles of order 1 at zeros and poles of f
residue of
order of zero or ngative of order of pole

By residue theorem

$$\frac{1}{2\pi i}\int_{\delta D} \frac{f'(z)}{f(z)} = \Sigma \text{ order of zeros} - \Sigma \text{ order of poles}$$

Note: more generally we replace $D \subseteq \Omega$ by a s.c domain $\Delta \subseteq \Omega$ s.t $\delta\Delta$ is rectifiable eg: $\Delta =$  $S\Delta \underset{n}{\approx} 0$

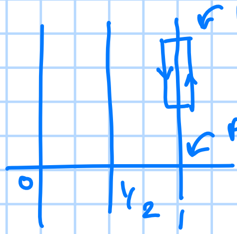
$\Delta =$  ,  $= \Delta$

motivation/application:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \text{for } Re(s) > 1$$

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

and $\Lambda(1-s) = \Lambda(s)$



Argument principle

pole at $\zeta(s)$ for $s=1$

Riemann Hypothesis:

All "non trivial" zeroes of $\zeta(s)$ satisfies $Re(s) = \frac{1}{2}$

9th April :

Argument principle :

If f is a meromorphic function on Ω and disk $D \subseteq \Omega$ s.t f is nonzero and continious on ∂D then:

$$\frac{1}{2\pi i}\int_{\delta D} \frac{f'}{f}(z)\,dz = \sum_{\text{zeroes}} \text{order of zeroes} - \sum_{\text{poles}} \text{order of poles}$$

in D

more genally, if γ is a path in Ω and f is like before s.t f is non-zero and continious on $\{\gamma\}$ then:

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'}{f}(z)\,dz = \sum_{\substack{\text{zeroes}\\ z_k}} n(z_k,\gamma)\times \text{order}(z_k) - \sum_{\substack{\text{poles}\\ p_k}} n(p_k,\gamma)\times \text{order}(p_k)$$

Note : proof of both can be done using residue theorm, and γ is not needed to be s.c curve.

Theorm : (Rouche's theorm) suppose f and g are two hol funtions in open set $\Omega \subseteq \mathbb{C}$, and $\exists$ D, a disk s.t $D \subseteq \Omega$ and f does not vanish on δD

if $|f(z)| > |g(z)|$ $\forall z \in \delta D$ then
# of zeroes of f inside int(D) = # zeroes of $f+g$ inside int(D)

(A small perturbation does not change the no. of zeroes)
→ g is our small perturbation

proof : consider $h_t(z) = f(z) + t g(z)$, $0 \leq t \leq 1$
now h_t is hol as linear comb of f and g
$\Rightarrow h_t \in H(\Omega)$

now, $h_0(z) = f(z)$
$h_1(z) = f(z) + g(z)$

also $|f(z)| > |g(z)| \Rightarrow |f(z)| > t|g(z)|$
$\Rightarrow |f(z) - t\,g(z)| > 0$ $\forall$ $t \in [0,1]$, $z \in \delta D$

so $|f(z) + t g(z)| > 0$ $\forall t \in [0,1]$, $z \in \delta D$

so $f(z) + tg(z)$ is non vanishing (on δD)

now applying argument priniple to $h_t(z)$ (as deiominator is non van)

n_t = # of zeroes of $h_t(z)$ inside int(D)

$$\Rightarrow n_t = \frac{1}{2\pi i}\int_{\delta D} \frac{h_t'}{h_t}(z)\,dz$$

this is an integer

now, $$n_t = \frac{1}{2\pi i}\int_{\delta D} \frac{f'(z) + t g'(z)}{f(z) + tg(z)}\,dz$$

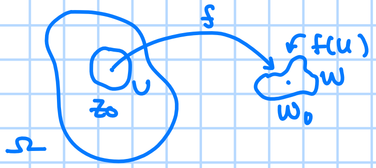
now n_t is cont as
$\lim_{h\to 0} n_{t+h} = n_t$

as $\operatorname{var}(\delta D) < \infty$ (or this method to show n_t is cont)
and integrand is hol & uniform cont as funtion of t (δD is compact)

$\Rightarrow n_t$ is integr and cont
$\Rightarrow n_t =$ constant
$\Rightarrow n_0 = n_1$
$\Rightarrow$ # of zeroes of f in $\operatorname{int}(D)$
$=$ # of zeroes of $f+g$ in $\operatorname{Int}(D)$

Theorm: (open mapping theorm) If $\Omega \underset{open}{\subseteq} \mathbb{C}$ and f is non-constant and hol on Ω then f is an open map. i.e if $U \underset{open}{\subseteq} \Omega$, then $f(U) \underset{open}{\subseteq} \mathbb{C}$

proof: for $z_0 \in \Omega$, let $w_0 = f(z_0)$
now, we want to show that, $\exists \delta > 0$ s.t
$|w - w_0| < \delta \Rightarrow w = f(z)$



let $g(z) = f(z) - w = f(z) - w_0 + w_0 - w$
choose $\delta > 0$ s.t $D(z_0, \delta) \subseteq \Omega$
domain

and $f(z) \neq w_0 \;\; \forall z \in \delta D$
as if we are never able to choose such disk
then for all disk, $\exists$ a point st $f(z) = w_0$ ← we find a continious path
$\Rightarrow f \equiv w_0$ but f is non-constant

now $|f(z) - w_0|$ is non vanishing on δD
and so $|f(z) - w_0|$ has a minima on δD
say $\varepsilon > 0$ s.t
$|f(z) - w_0| > \varepsilon$

choose w s.t $|w - w_0| < \varepsilon$
then
$F(z) = f(z) - w_0$
$G(z) = w_0 - w$
then $|f(z) - w_0| > |w_0 - w|$ on δD
as $|f(z) - w| > \varepsilon$ & $|w_0 - w| <$
$\Rightarrow$ # of zeroes of F = # of zeros on $F+G$

$\Rightarrow F + G = g$, $F(z) = f(z) - w_0$ has atleast one zero

$\Rightarrow g(z)$ has atleast one zero inside δD (By Rouche)
$\Rightarrow \exists z \in \operatorname{int}(D)$ s.t
$g(z) = 0$
$\Rightarrow f(z) = w$

Theorm: (maximum modulus priciple) If $\Omega \underset{open}{\subseteq} \mathbb{C}$ and f is holomorphic on Ω, then f cannot attain its maximum on Ω.

proof: f attains max (i.e $|f|$ is max) at some $z_0 \in \Omega$, then choose a small disk $D(z_0, r) \subseteq \Omega$

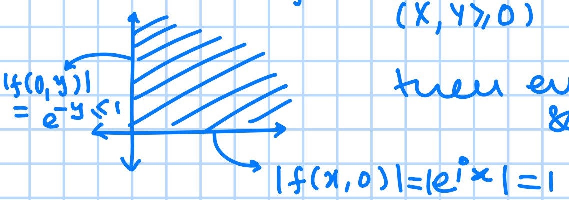
cosider $f(\operatorname{int} D(z_0, r))$ which is open ($\because f$ is open map)
but $\exists\, w \in f(\operatorname{int} D(z_0, r))$
s.t $|w| > |f(z_0)|$
and $w = f(z)$
for $z \neq z_0 \Rightarrow$ this is a contradiction

Corr: If Ω is compact, then $\sup_{z \in \text{int}(\Omega)} |f(z)| \leq \sup_{z \in \bar{\Omega} \setminus \text{int}(\Omega)} |f(z)|$

$\bar{\Omega} \setminus \text{int}(\Omega) = \partial\Omega$

Note: compactness is necessary

Eg: $f(z) = e^{iz}$ for closed Ω = first quadrant $(x, y \geq 0)$



then exceed it

so Ω has to be compact

Now, our next cases:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$ $\leftarrow$ $\sin(z)$ as product of zeroes

11^{th} April :

Recap : Argument principle, Rouche's theorem, open mapping theorem, max modulus priciple (this all derives from Cauchy)

Basel problem :

Euler : $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$

we will be defining infinite product and their convergence

defn : (Infinite product) Given $\{b_n\}_{n\geq 1}$, $b_n \in \mathbb{C}$, say that $\prod_{n=1}^{\infty} b_n$ converges if $\{\prod_{n=1}^{N} b_n\}_{N\geq 1}$ converges ($\forall \varepsilon>0, \exists N$ definition)

Recall that $\sin(\pi z)$ vanishes exactly at $0, \pm 1, \pm 2, \ldots$, so how close are $\sin(\pi z)$ and $\pi z \prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)$ as N grows

Note : $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \quad \forall z \in \mathbb{C}$ (we will prove this today and in tutorial prob.)

we use that ∞-product converges uniformly on compact sets (we will prove this also)

Now, $\pi z \prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)$ where $|z|<1$

$$= \pi z (1-z^2)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots$$

$$= \pi z - \pi z^3 \underbrace{\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]}_{\text{we want to find this}} + \cdots$$

(so if $\pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \sin \pi z$ we get $\sum \frac{1}{n^2} = \frac{\pi^2}{3!}$)

we also have $\sin \pi z = \pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z^5)}{5!} + \cdots$

$$\frac{\pi^3}{3!} = \pi\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

similary, we can find $\sum \frac{1}{n^4}$ by z^5 coefficient,

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Note : $\sum \frac{1}{n^3} = \alpha$ is irrational, it is known as aperi's constant but we don't know the value.

$$\sum \frac{1}{n^3} = \alpha = \zeta(3)$$ ← zeta funtion of 3

(we also don't know if α is algebric or trasidental)

proposition : If $\{a_n\}_{n\geq 1}$ is a sequence of complex numbers s.t $\sum_{n\geq 1} |a_n| < \infty$, then $\prod_{n=1}^{\infty}(1+a_n)$ converges (absolute conv)

proof : $\sum_{n\geq 1} |a_n|$ converges as $|a_n| \to 0$ as $n\to\infty$
$\Rightarrow \exists N \in \mathbb{N}$ s.t
$\forall n>N \Rightarrow |a_n| < \frac{1}{2}$ (after a N, $|a_n| < \frac{1}{2}\ \forall n>N$)

$\therefore$ for any M : $\prod_{n=1}^{M}(1+a_n) = \underbrace{\prod_{n=1}^{N}(1+a_n)}_{\text{finite}} \prod_{n=N+1}^{M}(1+a_n)$

(wlog $M>N$)

for $n>N$, $1+a_n = e^{\log(1+a_n)}$ (as $1+a_n \neq 0$) $|1+a_n|>0$ ← Always

$$\prod_{n=1}^{M}(1+a_n) = \prod_{n=1}^{N}(1+a_n)\underbrace{\prod_{n=N+1}^{M} e^{\log(1+a_n)}}$$

$$= e^{\sum \log(1+a_n)}$$

let $B_{N,M} = \sum_{n=N+1}^{M} \log(1+a_n)$

now, $|\log(1+z)| \leq 2|z|$ for $|z| \leq \frac{1}{2}$ (in tutorial, from Cauchy's formula by derivative bound)

$\Rightarrow \underbrace{|\log(1+a_n)|}_{\text{sum finite as } \sum |a_n| \text{ is finite}} \leq 2|a_n|$

$\Rightarrow \sum_{n=N+1}^{\infty} |\log(1+a_n)|$ is finite

$\Rightarrow \lim_{M\to\infty} B_{N,M} < \infty$

say $B_N := \lim_{M\to\infty} B_{N,M}$

$\Rightarrow \prod_{n=1}^{\infty}(1+a_n) = \underbrace{\prod_{n=0}^{N}(1+a_n)}_{\text{convergent}} \underbrace{e^{B_N}}_{\text{convergent}}$

now $\prod_{n=1}^{\infty}(1+a_n)$ converges, moreover

$\prod_{n=1}^{\infty}(1+a_n) = 0 \Leftrightarrow \exists\, n \in \mathbb{N}$ s.t $(1+a_n) = 0$, as e^{B_N} can never be 0.

proposition: If $\{F_n(z)\}_{n\geq 1}$ is a sequence of hol functions. $F_n: \Omega_{open} \to \mathbb{C}$. say $\exists\ \{c_n\}_{n\geq 1}$ s.t $c_n > 0$ s.t $\sum_{n=1}^{\infty} c_n < \infty$ and $|1-F_n(z)| < c_n\ \forall n \geq 1,\ \forall z \in \Omega$

then i) $\prod_{n\geq 1} F_n(z)$ converges uniformly on Ω (domain) to a hol function $F(z) = \prod_{n=1}^{\infty} F_n(z)$

ii) If each $F_n(z)$ are non vanishing then:

$$\frac{F'(z)}{F(z)} = \sum_{n\geq 1} \frac{F_n'(z)}{F_n(z)}$$

proof: lets define $a_n(z) = F_n(z) - 1$

then $|a_n(z)| < c_n\ \forall n \geq 1$

and $\sum c_n < \infty$

$\Rightarrow$ by weistrass - M-test :

$\sum_{n=1}^{\infty} |a_n(z)| < \infty$

$\downarrow$

converges for $\forall z \in \Omega$

moreover by M-test the convergence is uniform.

$\Rightarrow \prod_{n=1}^{\infty}(1+a_n)$ converges $\forall z \in \Omega$ (uniform conv needed for $\prod(1+a_n)$)

$\Rightarrow \prod_{n=1}^{\infty} F_n(z)$ converges $\forall z \in \Omega$

to show uniform convergence, make two parts, the second part as $e^{B_{N,M}(z)}$ and the first part is finite

$\longrightarrow$ converges uniformly

$\Rightarrow \prod_{n=1}^{\infty} F_n(z)$ is hol on Ω as we wrote it as $\prod F_n(z) e^{B_N} = F(z)$

e^{B_N} is hol

and $\prod F_n(z)$ is hol

for second part, $G_N(z) = \prod_{n=1}^{N} F_n(z)$, we have shown that

$$\frac{G_N'(z)}{G_N(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$$ Non-vanishing

$$\Rightarrow \frac{G_N'(z)}{G_N(z)} \text{ is holomorphic}$$

given $z \in \Omega$, choose compact set $K \subseteq \Omega$ s.t $z \in K$

since $F_n(z)$ are non-vanishing

K is compact $\Rightarrow \exists\, \varepsilon_n > 0$ s.t

$$|F_n(z)| \geq \varepsilon_n \;\; \forall z \in K$$

$$\Rightarrow |G_N(z)| \geq \prod_{n=1}^{N} \varepsilon_n$$

$$\Rightarrow \frac{1}{|G_N|} \leq \frac{1}{\prod_{n=1}^{N} \varepsilon_n}$$

now as G_N converges uniformly on K as $n \to \infty$

$\Rightarrow G_N'$ converges uniformly on K $n \to \infty$ (theorm done before)

$\Rightarrow \frac{G_N'}{G_N}$ converges on K $\left(\because \frac{1}{|G_N|} \leq \varepsilon \right)$

$$\Rightarrow \lim_{N \to \infty} \frac{G_N'(z)}{G_N(z)} = \frac{F'(z)}{F(z)}$$

$$= \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$$

Weierstrass product formula:

given $\{a_n\}$ a sequence of $\mathbb{C}$-nos s.t $|a_n| \longrightarrow \infty$ then $\exists$ a hol fn $f: \mathbb{C} \longrightarrow \mathbb{C}$ s.t fn vanishes at exactly $z = a_n$ (with prescribed order of vanishing)

If f_1 & f_2 are two such functions, then $\exists\, g: \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic s.t

$$f_1(z) = e^{g(z)} f_2(z)$$

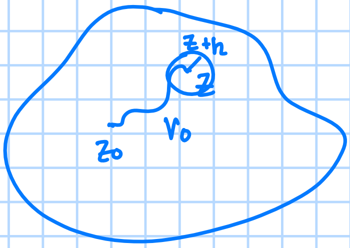
lemma: If f is holomorphic and non-vanishing on Ω, and Ω is s.c then $\exists\, g: \Omega \to \mathbb{C}$ hol s.t

$$f(z) = e^{g(z)} \;\; \forall z \in \Omega$$

proof: Fix $z_0 \in \Omega$ and define $g(z) = \int_\gamma \frac{f'(w)}{f(w)} dw + c_0$

γ is a path $z_0 \to z$ where $c_0 = \log(f(z_0))$ (Priniple domain)
now as Ω is s.c γ does not matter

Then $g(z)$ is holomorphic by $\lim\limits_{h\to 0} \frac{g(z+h)-g(z)}{h}$ exist



and $\lim\limits_{h\to 0} \frac{g(z+h)-g(z)}{h} = g'(z) = \frac{f'(z)}{f(z)}$

also now $\frac{d}{dz}(f(z)e^{-g(z)})$

$= f'(z)e^{-g(z)} - f(z)\,g'(z)\,e^{-g(z)}$

$= f(z)e^{-g(z)}\left[\frac{f'(z)}{f(z)} - g'(z)\right]$

$= 0$

$\Rightarrow f(z)e^{-g(z)} = c$

$\Rightarrow f(z) = ce^{+g(z)}$

now at $z = z_0$

$\Rightarrow f(z_0) = ce^{+g(z_0)}$

$= ce^{+c_0}$
$= ce^{+\log(f(z_0))}$

$\Rightarrow f(z_0) = c\, f(z_0)$
$\Rightarrow c = 1$

So, $f(z) = e^{g(z)}$

principle part

$\therefore \exists$ hol $g(z) = \int_\gamma \frac{f'(z)}{f(z)} + \log(f(z_0))$

s.t $f(z) = e^{g(z)}$

Theorm: $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ are both entire (hol on $\mathbb{C}$) with same prescribed zeroes $\{a_n\}$ then $h(z) = \frac{f_1(z)}{f_2(z)}$ satisfies $h(z) \neq 0$ when $z \neq a_n$ and at $z = a_n$ $h(z)$ has removable singularity and $\lim\limits_{z\to a_n} h(z) \neq 0$

and so $\exists\, g(z)$ s.t $f_1(z) = e^{g(z)} f_2(z)$, $g(z)$ is hol : $\mathbb{C} \to \mathbb{C}$

proof: in $D(a_n, r_n)$ for small r_n, we can write $f_1(z) = (z-a_n)^{m_n} h_1(z)$
$f_2(z) = (z-a_n)^{m_n} h_2(z)$

where $h_1, h_2 \neq 0$ on $D(a_n, r_n)$

$\Rightarrow \frac{f_1(z)}{f_2(z)} = \frac{h_1(z)}{h_2(z)}$ on $D(a_n, r_n)\setminus\{a_n\}$

$\Rightarrow \frac{f_1(z)}{f_2(z)} = \frac{h_1(z)}{h_2(z)}$ on $D(a_n, r_n)$

So, $\frac{f_1}{f_2}$ is entire non-vanishing fuction

$\Rightarrow \exists\, g(z) : \mathbb{C} \to \mathbb{C}$ & entire s.t (from prev proof, such $g(z)$ exist)

$\frac{f_1(z)}{f_2(z)} = e^{g(z)}$

$$\Rightarrow \quad f_1(z) = e^{g(z)} f_2(z)$$

15th April :

Weistrass product formula :

given a sequence of $\mathbb{C}$-nos $\{a_n\}_{n\geq 1}$ s.t $\lim_{n\to\infty} |a_n| = \infty$, then there exist a hol function $f: \mathbb{C} \to \mathbb{C}$ s.t f vanishes at $z = a_n \ \forall n \geq 1$ to desired multiplicity

moreover, if f_1 and f_2 are two such functions then $\exists g: \mathbb{C} \to \mathbb{C}$ holomorphic s.t $f_1 = f_2 e^g$ (so we would have to make this f and then find its family using e^g)

Recap : we showed existance of $g: \mathbb{C} \to \mathbb{C}$ holomorphic s.t $f_1 = e^g f_2$ ($\frac{f_1}{f_2}$ is holomorphic)

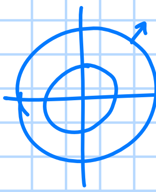
Remark : if $\lim_{n\to\infty} |a_n| \neq \infty$ then $\exists$ an infinite subsequence which is borded $\{a_{n_k}\} \subset \{a_n\}$ and so by bolzano wiestrass every bounded seq has converging subsequence

$\Rightarrow$ limit point of zeroes in $\mathbb{C}$

$\Rightarrow$ By theorm covered before, the hol function becomes constant

can we have an entire function with uncoutable many 0?

lets say that if f has coutable many zeroes then each $B(0,n)$ has coutable many zeroes



B(0,n)

if f has uncotable zeroes then

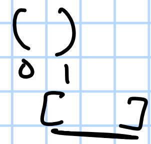
$\exists n \in \mathbb{N}$ s.t

$B(0,n)$ has uncotable zeroes

$\Rightarrow \exists$ a bounded seq

$\Rightarrow \exists$ a convergent subsequence

$\Rightarrow f$ becomes 0 everywhere



Construction of $f_1 = e^g f_2$:

Naive construction:

consider:

$$z^{m_0} \prod_{\substack{n\geq 1 \\ a_n \neq 0}} \left(1 - \frac{z}{a_n}\right)^{m_n}$$

but this may not converge

fix: if $p(z)$ is a polynomial then

$$\lim_{\substack{z\to\infty \\ x>0,\, z = x+iy}} |P(z) e^{-z}| = \lim_{\substack{z\to\infty \\ x>0}} |P(z)| e^{-x} \quad (\text{letting } x\to\infty)$$
$$= 0$$

construction of $f: \mathbb{C} \to \mathbb{C}$ with priscribed 0s

define canonical factor:

$$E_0(z) = 1 - z$$
$$E_n(z) = (1-z) e^{z + z^2/2 + \cdots z^n/n} \quad \text{for } n \geq 1$$

intuition for this is as $|z| < 1$

then $\underbrace{\log(1-z)}_{\text{converges}} = -\left(z + \frac{z^2}{2} + \cdots\right)$

so $(1-z)e^{\underbrace{z+z^2/2+\cdots z^n/n}_{-\log(1-z)}} \approx 1$ when $|z|<1$

$\left(\because (1-z)e^{-\log(1-z)} = 1 \right)$

therefore $f(z) = z^{m_0} \cdot \prod_{n=1}^{\infty} E_n((z/a_n))$

satisfies the condition

with a_n are repeated multiplicity many times

we are repeating multiplicities

i.e $a_1 = i$

$a_2 = i$

then $E_1(z/i) \times E_2(z/i)$ in fuction

<u>lemma</u>: If $|z| \leq \frac{1}{2}$ then $|1-E_k(z)| \leq c|z|^{k+1}$ for const $c>0$ independent of k

<u>proof</u>: as $|z| \leq \frac{1}{2}$ $(1-z) \neq 0$

and

$(1-z) = exp(\log(1-z))$ Pricide argument

$E_k(z) = e^{\log(1-z)+z+\cdots z^k/k}$

$= e^{-\left(\underbrace{\frac{z^{k+1}}{k+1} + \frac{z^{k+2}}{k+2} + \cdots}_{\text{Reminder converges}}\right)}$

$= e^{w}$

now, $|w| = \left| \sum_{n=k+1}^{\infty} \frac{z^n}{n} \right|$

$= |z|^{k+1} \left| \sum_{n=0}^{\infty} \frac{z^n}{n+k+1} \right|$

$\leq |z|^{k+1} \sum_{n=0}^{\infty} |z|^n$

$\leq |z|^{k+1} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

$= 2|z|^{k+1}$

$|w| \leq 2|z|^{k+1}$ (showing $|w| \leq c|z|^{k+1}$)

now, $|1-E_k(z)| = |1-e^w|$

$= |\cancel{1} - (\cancel{1} + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots)|$

$= \left| w + \frac{w^2}{2!} + \cdots \right|$

$= |w| c_1$

depends on $2|z|^{k+1}$

but $|z| \leq \frac{1}{2}$ so goes to 0

$\Rightarrow \leq \underbrace{2 \cdot c_1}_{c} |z|^{k+1}$

so, $|1-E_k(z)| \leq c|z|^{k+1}$

Theorem: The constructed infinite product converges

proof: Choose $R>0$, divide $\{a_n\}$ into whether $\underbrace{|a_n|\leq 2R}_{\text{finite many}}$ or $\underbrace{|a_n|>2R}_{\text{infinite}}$

then $z^{m_0}\prod_{n=1}^{\infty} E_n(z/a_n)$

$$= z^{m_0} \underbrace{\prod_{|a_n|\leq 2R} E_n(z/a_n)}_{\text{converges and hol as all terms are finite}} \underbrace{\prod_{|a_n|>2R} E_n(z/a_n)}_{\text{we only have to care about this convergence}}$$

(we are choosing R)

for convergence of $\prod_{|a_n|>2R} E_n(z/a_n)$ recall, lemma/proposition that

Bounded by this

$$|1-E_n(z/a_n)| \leq C\left|\frac{z}{a_n}\right|^{n+1} \leq \frac{C}{(2)^{n+1}}$$

choose R to be $>|z|$ (choose $R>|z|$ given $z\in\mathbb{C}$)

i.e

$|a_n|>2R \Rightarrow |a_n|>2|z|$

$\Rightarrow \frac{1}{2} > \frac{|z|}{|a_n|}$

so inside $B(0,R)$ the ∞-product converges uniformly from proposition done previously

$\therefore z^{m_0}\prod_{|a_n|\leq 2R} E_n(z/a_n) \prod_{|a_n|>2R} E_n(z/a_n)$ is holomorphic

this proves that ∞-product conv to a hol function, moreover zeroes are exactly at $z=a_n$ with multiplicity 1.

Note: $f(z) = z^m \prod_{n\geq 1} E_n(z/a_n)\cdot e^{g(z)}$

Corr: Every meromorphic function is a ratio of two hol functions

we have done the local version of this as $z_0\in\mathbb{C}$, $\exists$ U open $\ni z_0$ s.t $f(z) = \frac{g(z)}{h(z)}$ $\forall z\in U$

If pole at z_0 then $f(z) = \underbrace{(z-z_0)^{-n}}_{h(z)} g(z)$ for some $n\geq 1$

proof: say $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are sets of zeros and poles of $f(z)$

then construct $h(z)$ with zeroes exactly at $\{b_n\}_{n\geq 1}$ with order of zero = order of pole (we generate h using canonical forms)

then $f\cdot h$ has only removable singularities at $z=b_n$

$\Rightarrow f\cdot h = g(z)$ some holomorphic function which is entire

$\Rightarrow f = \frac{g}{h}$ for g,h

If $g,h:\mathbb{C}\longrightarrow\mathbb{C}$ are hol then: $f=\frac{g}{h}$ is meromorphic with poles of f at zeroes of h with same multiplicity. (so meromorphic functions are ratio of holomorphic)